VIBRATION ANALYSIS OF ISOTROPIC AND ORTHOTROPIC PLATES AND ROTATING DISCS USING FINITE ELEMENT METHOD

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DEDICATED

TO

MY MOTHER

CERTIFICATE

This is to certify that the thesis entitled VIBRATION ANALYSIS OF ISOTROPIC AND ORTHOTROPIC PLATES AND ROTATING DISCS USING FINITE ELEMENT METHOD, by Mr. N. Shriranga Achar has been carried out under my supervision and has not been submitted elsewhere for a degree.

November, 1934

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NOMENCLATURE

a	Outer radius of the element
a_1, a_2, a_3 and a_4	Constants
b	Inner radius of the element
C	Constant , 2π for $m = o$ π for $m \geq 1$
$D_r, D_\theta, D_{r\theta}$, D_x, D_y, D_{xy}	Regidity ratios of orthotropic material in principal directions of orthotropy in polar and cartesian co-ordinate systems respectively
E	Young's modulus
G	Shear modulus
h	Thickness of the plate
h_1	Thickness at the inner edge of the annular plate
h_2	Thickness at the outer edge of the annular plate
$[K]$	Stiffness matrix
l	Length of the element
L_x	Half the width of rectangular element
L_y	Half the breadth of rectangular element
$[M]$	Mass matrix
m	Number of nodal diameters
n	Number of nodal circles

$\lfloor N \rfloor$	Vector containing the shape functions
Ne	Number of element
R_1	Inner radius of annular plate
R_2	Outer radius of annular plate
T	Kinetic energy
U	Strain energy
u	Displacement in radial direction
W	Displacement in lateral direction
W_J	Displacement in lateral direction at node J
W_{S_J}	Derivative of W w.r.t. S at node J
$\dot{W}$	Velocity in lateral direction
$\ddot{W}$	Acceleration in lateral direction
$\{W\}^{(ne)}$	Vector containing the nodal degrees of freedom of an element
$\lfloor \varepsilon \rfloor$	Strain matrix
ρ	Mass density
ν	Poisson's ratio
λ	Natural frequency
σ	Stress
Ω	Angular velocity
α	Frequency parameter
B-C	Boundary conditions
S-C	Simple-clamped
F-C	Free-clamped
C-C	Clamped-clamped

First boundary condition refers to outer edge
Second boundary condition refers to inner edge

ABSTRACT

The vibration analysis of various types of plates namely rectangular (with and without cut outs), annular, annular sector, circular and rotating discs has been done. For the case of rectangular plate, a four noded rectangular element with four degrees of freedom at each node, namely lateral displacement, two bending rotations about two co-ordinate axes respectively and twist has been used.

A circular sector element with same number of degrees of freedom and an annular element (semianalytical method) have been used in case of circular plates. The annular element takes into consideration the thickness variation in the radial direction. The vibration analysis of rotating discs has been done using the semianalytical method.

Nowadays the higher stiffness to weight ratio of orthotropic plates is drawing much attention. Hence throughout the formulation and analysis of results, orthotropic plates are given emphasis.

CHAPTER - I

INTRODUCTION

1.1 GENERAL

One of the commonly used structural component in the industrialised world is a plate. In many of the design problem specifications merely ensuring that plates will withstand the applied static loads will prove to be inadequate since a variety of structures used in land, sea, air and space are subjected to dynamic stresses and displacements induced by periodic forces acting on the lateral surface of the plate. Random forces are expected, for example, on the surfaces of the plates exposed to tangential gas flow as in aircraft components or stationary structures exposed to high wind velocity. Such forces can also be expected when a fluid flow takes place along the plates as in the case of rectangular plates used as the hull of a ship.

Periodic forces are likely to be experienced when plates form a part of the rotating machinery operating at a specified speed. To ensure mechanical integrity it is also necessary to predict the influence of stresses

in the discs which are due to the centrifugal force arising in the rotating disc due to its own mass. Turbomachines and superchargers are typical examples where such rotating discs are used.

Usually little can be done to change the nature of the driving forces. Hence the most commonly used design technique for solving such problems where matching of the driving forces and natural frequencies is to be done is to alter the plate geometry or boundary conditions, so that the natural frequency zone is away from the frequency zone of the driving energy and hence the resonance is avoided. The importance of the knowledge of the natural frequencies of plates in such applications is obvious.

1.2 LITERATURE SURVEY

In literature one finds an extensive treatment of the problems on the flexural vibration of isotropic plates - both rectangular and circular. The problem of vibration of isotropic plates having free edges was investigated long back by Kirchhoff, Lamb and Rayleigh using Poisson-Kirchhoff theory. Timshenko [1] used energy method for solving the case of isotropic plate with clamped edges. Thein Wah [2] investigated the case of simply supported edge using Poisson-Kirchhoff

theory for the basic formulation. Numerical results are also given for the frequency parameter, for modes of vibration consisting of nodal circles and diameters. An extensive study of uniform annular isotropic plates has been done by Vogel et al. [3]. Here the classical theory of flexural motions of elastic plates was used to determine the natural frequencies for various boundary conditions of annular plates. An investigation was also made into the effects of Poisson's ratio on frequency parameters by taking different values of Poisson's ratio. Its effect has been found to be insignificant on natural frequencies. Hence in reference [3] the analysis has been done taking the value of Poisson's ratio as 0.3.

Blevins [4] has given numerical results for rectangular, annular and circular plates for various boundary conditions. Zinckiewicz [5] has given applications of finite element method to the vibration analysis of plates. The common elements used are triangular and rectangular [6] ones. Triangular elements are more useful since they can be used to represent curved boundaries.

Ergutoudis [7] has developed quadrilateral element with curved sides. Olson et al. [8] have developed two plate bending finite elements in polar coordinates to study the vibration of circular plates.

The first element has three nodal corners and is in the shape of the sector of a circle. The second element has four nodal corners and forms an annular sector. The transverse displacement and two rotations about the two co-ordinate axes respectively are the three degrees of freedom used at each node in both the cases. Non-dimensional frequency parameter for a circular plate with clamped, free and simply supported boundary conditions are tabulated and compared with the exact values. Kirkhope et al. [9] have used semianalytical method to study the dynamic behaviour of isotropic circular plates and rotating discs for various boundary conditions. Kultar Singh [10] and Benjamin [11] have also used the annular element to study the dynamic behaviour of plates.

Even though the literature contains various types of solution techniques of vibration analysis of isotropic plates,that of orthotropic plates needs some more attention. Kirmser et. al. [12], Pandalai et al. [13], Minkarah et al. [14], Joung [15], Bellini [16] Huang [17] and Nowinski et al. [18] have attempted some problems on orthotropic bodies subjected to dynamic loading. The existance of material singularity has been reported by many of them. This is due to fact that circumferential and radial modulai can not be different at the centre. In order to overcome the above problem

some have assumed an isotropic core of small radius at the centre.

Ramaiah and Vijayakumar [19] have found out the lowest eigen values for various boundary conditions using Rayleigh-Ritz method with simple polynomials as admissible functions. In an attempt to find out the natural frequencies of higher modes of vibration, Ramaiah et al. [20] have used the classical Rayleigh-Ritz method by introducing the co-ordinate transformation. They have obtained the natural frequencies corresponding to the asymmetric modes (modes with one nodal circle and two nodal diameters). However application of this method is generally cumbersome since the method involves solution of a large set of simultaneous equations for higher accuracy of results. They have given the results for free-free, clamped-free and simply supported - free annular plates of radii ratio 0.5 and 0.1. Regarding the vibration of rotating discs Evessham's [21] work can be mentioned here where a series solution has been obtained.

1.3 PRESENT WORK

The study of nature of the natural frequencies of plates (mainly orthotropic) for various boundary conditions with varying thickness and rotating discs is the aim of the present work. Using rectangular and

annular sector elements (described in Chapter II) vibration analysis of plates with geometrical nonlinearity like plates with cut-outs can be made. It can also take into account different types of loading.

For plates where there is linearity in material property as well as symmetry in geometry semianalytical method is more efficient reducing the problem to a one dimensional one. In the present work vibration analysis of orthotropic plates with varying thickness and rotating orthotropic discs has been done using the above method (Chapter III and IV). For uniform thickness plates good results are obtained with few elements (2 to 4). But slightly more number of elements (8 to 10) are required for plates with varying thickness.

CHAPTER - II

VIBRATION ANALYSIS OF RECTANGULAR AND ANNULAR PLATES

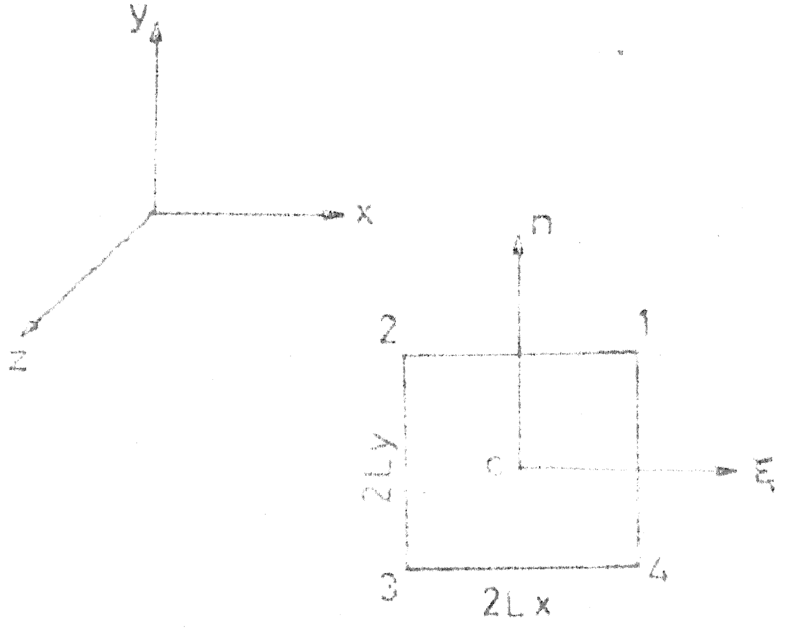
2.1 INTRODUCTION

For vibration analysis of rectangular plates, rectangular elements have been used. A typical rectangular element is shown in Fig. 1. The element chosen is a conforming element with four degrees of freedom per node, they being deflection W, two rotations W_x , W_y and twist W_{xy}. This rectangular element with these sixteen degrees of freedom is compatible in displacement and its first derivatives i.e. it is a C1 element. For circular plates an annular sector element, shown in Fig. 2 has been chosen with same sixteen nodal degrees of freedom in polar co-ordinates.

2.2 ASSUMPTIONS

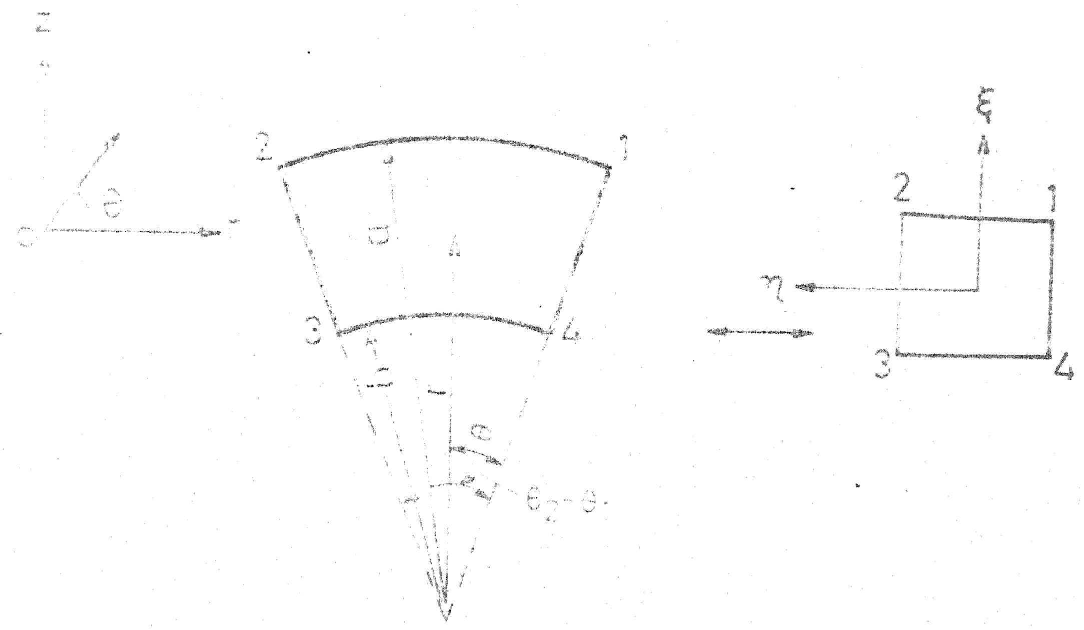
The present formulation is based on the thin plate theory (for small deflections) which makes the following assumptions [22].

1. The plates deform through flexural deformation only.



Nodal degrees of freedom are W_i, W_{x_i}, W_{y_i} & W_{xy_i}, i = 1,4

Fig.1 Rectangular element



Nodal degrees of freedom are W_i, W_{r_i}, W_{θ_i} & $W_{r\theta_i}$, i=1,4

Fig.2 Sector element

2. Deformations are small in comparison with the plate thickness.

3. Normals to the mid-surface of the undeformed plate remain straight and normal to the middle plane during bending.

4. Rotary inertia and shear deformation are negligible.

5. The inplane stresses (membrane stresses) are negligible.

2.3 FORMULATION OF THE PROBLEM

Following formulation is easily available in literature and is given here for easy reference. For an elastic linear material strain energy is

$$U = \frac{1}{2} \int_V \lfloor \varepsilon \rfloor \; [D'] \; \{\varepsilon\} \; dv \qquad (2.1)$$

and the kinetic energy is

$$T = \frac{1}{2} \int_A \rho \, h \; \dot{W}^2 \; dA \qquad (2.2)$$

Assuming $W^{(e)} = \lfloor N \rfloor \; \{W\}^{(ne)}$ (2.3a)

Using eqn. (2.3a) $\lfloor \varepsilon \rfloor = [B'] \; \{W\}^{(ne)}$ (2.3b)

$$U^{(e)} = \frac{1}{2} \lfloor W \rfloor^{(ne)} \int_V [B']^T \, [D'] \, [B'] \; dv \; \{W\}^{(ne)} \qquad \ldots\ldots \quad (2.4)$$

$$T^{(e)} = \frac{1}{2} \lfloor \dot{W} \rfloor^{(ne)} \int_A \rho \, h \; \{N\} \, [N] \; dA \; \{\dot{W}\}^{(ne)} \qquad \ldots\ldots \quad (2.5)$$

Now applying the Hamilton principle

$$\partial \int_{t_1}^{t_2} (T - U)\, dt = 0 \qquad (2.6)$$

Substituting eqns. (2.4) and (2.5) in eqn. (2.6) one gets

$$[M]^{(e)} \{\ddot{W}\}^{(ne)} + [K]^{(e)} \{W\}^{(ne)} = \{0\} \qquad \ldots \qquad (2.7a)$$

where $[M]^{(e)} = \int_A \rho h \{N\} \lfloor N \rfloor \, dA$ and

$$[K]^{(e)} = \int_V [B']^T [D'] [B'] \, dv \qquad (2.7b)$$

Assuming harmonic motion, eqn. (2.7) becomes

$$([K]^{(e)} - \lambda^2 [M]^{(e)}) \{W\}^{(ne)} = \{0\} \qquad (2.8)$$

which are the equations of a motion for any element.

2.3.1 Rectangular Plate

For a rectangular plate strain matrix $[\varepsilon]$ is [22]

$$\lfloor \varepsilon \rfloor = \lfloor \varepsilon_x \quad \varepsilon_y \quad \gamma_{xy} \rfloor$$

$$= \lfloor -z\frac{\partial^2 \bar{w}}{\partial x^2} \quad -z\frac{\partial^2 \bar{w}}{\partial y^2} \quad -2z\frac{\partial^2 \bar{w}}{\partial x \partial y} \rfloor \qquad \ldots \qquad (2.9)$$

Using eqn. (2.3) and eqn. (2.9), $[B']$ matrix for rectangular elements becomes

$$[B'] = Z \begin{bmatrix} -\lfloor N,_{xx} \rfloor \\ -\lfloor N,_{yy} \rfloor \\ -2\lfloor N,_{xy} \rfloor \end{bmatrix} = Z\,[B] \qquad (2.10)$$

Using eqn. (2.10), stiffness matrix $[K]^{(e)}$ of eqn. (2.7b) becomes

$$[K]^{(e)} = \iint [B]^T [D] [B]\, dx\, dy \qquad (2.11)$$

For orthotropic plates $[D]$ matrix is

$$[D] = \begin{bmatrix} D_x & D_1 & 0 \\ D_1 & D_y & 0 \\ 0 & 0 & D_{xy} \end{bmatrix} \qquad (2.12)$$

where $D_x = \dfrac{E_x\, h^3}{12(1-\nu_x\, \nu_y)}$

$D_y = \dfrac{E_y\, h^3}{12(1-\nu_x\, \nu_y)}$

$D_1 = D_x\, \nu_y \quad , \qquad D_{xy} = G_{xy}\, h^3/12$

Eqn. (2.12) reduces to

$$[D] = \frac{E\, h^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \qquad (2.13)$$

Consistent mass matrix $[M]^{(e)}$ of eqn. (2.7b) becomes

$$[M]^{(e)} = \iint \rho\, h \,\{N\}\, \lfloor N \rfloor\, dx\, dy \qquad (2.14)$$

2.3.2 Shape Functions

As said plate bending element needs the compatibility of displacements and slopes. The shape functions satisfying these conditions can be obtained from the following matrices product [23].

$$W^{(e)} = \lfloor N_{1x} \quad N_{2x} \quad N_{3x} \quad N_{4x} \rfloor \begin{bmatrix} W_3 & W_{y_3} & W_2 & W_{y_2} \\ W_{x_3} & W_{xy_3} & W_{x_2} & W_{xy_2} \\ W_4 & W_{y_4} & W_1 & W_{y_1} \\ W_{x_4} & W_{xy_4} & W_{x_1} & W_{xy_1} \end{bmatrix} \begin{Bmatrix} N_{1y} \\ N_{2y} \\ N_{3y} \\ N_{4y} \end{Bmatrix} \quad \ldots. \quad (2.15)$$

where N_{ix} and N_{iy} are the standard beam shape functions and are given in Appendix-A. Rearranging eqn. (2.15)

$$W^{(e)} = \lfloor N_1 \quad N_2 \ldots N_{16} \rfloor \{W\}^{(ne)} = \lfloor N \rfloor \{W\}^{(ne)} \quad \ldots. \quad (2.16)$$

where $\{W\}^{(ne)}$ is nodal displacement matrix (W_i , W_{x_i} , W_{y_i} , W_{xy_i} ; i = 1,4)

It may be noted that shape functions $\lfloor N \rfloor$ are represented interms of local co-ordinates ξ and η using the transformation

$$\xi = \frac{x - x_c}{L_x} \qquad \eta = \frac{y - y_c}{L_y} \qquad (2.17)$$

2.3.3 Annular Plates

For circular plates the strain matrix $\lfloor \varepsilon \rfloor$ from ref. [4] is

$$\lfloor \varepsilon \rfloor = \lfloor -z\frac{\partial^2 w}{\partial r^2} \quad -z\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) \quad -2z\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right) \rfloor \qquad \ldots \quad (2.18)$$

As said earlier sector element shown in Fig. 2 has been chosen for circular plates. Requirements of shape functions being same as earlier case,the shape functions used for rectangular plates can be used. Using eqns. (2.16) and (2.18),one gets from eqn. (2.3b)

$$[B'] = z \begin{bmatrix} -\lfloor N,_{rr} \rfloor \\ -\lfloor \frac{1}{r} N,_{r} + \frac{1}{r^2} N,_{\theta\theta} \rfloor \\ -2\lfloor \frac{1}{r} N,_{r\theta} - \frac{1}{r^2} N,_{\theta} \rfloor \end{bmatrix} = z\,[B] \qquad \ldots \quad (2.19)$$

Using eqn. (2.19), stiffness matrix $[K]^{(e)}$ of eqn.(2.7b) becomes

$$[K]^{(e)} = \iint [B]^T [D] [B]\, r\, dr\, d\theta \qquad (2.20)$$

where $[D]$ matrix for orthotropic plates is

$$[D] = \frac{h^3}{12(1-\nu_r \nu_\theta)} \begin{bmatrix} E_r & E_r \nu_\theta & 0 \\ E_r \nu_\theta & E_\theta & 0 \\ 0 & 0 & G_{r\theta}(1-\nu_r \nu_\theta) \end{bmatrix} \qquad \ldots \quad (2.21)$$

and for isotropic plates it becomes

$$[D] = \frac{E\,h^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \qquad (2.22)$$

Consistent mass matrix $[M]^{(e)}$ of eqn. (2.7b) becomes

$$[M]^{(e)} = \rho\, h \iint \{N\} [N]\, r\, dr\, d\theta \qquad (2.23)$$

In order to do the numerical integration, the sector element has to be mapped into a square of side 2 units. This is done using the following relations [25]

$$\xi = \frac{2\,(r - b)}{a - b} - 1 \quad \text{and} \quad \eta = \frac{2\,(\theta - \theta_1)}{(\theta_2 - \theta_1)} - 1 \qquad \ldots\ldots \quad (2.24)$$

From eqn. (2.24), r can be written as

$$r = \frac{1}{2}\,(1 + \xi)\,(a - b) + b \qquad (2.25)$$

2.3.4 Numerical Integration

For this Gauss Legendre quadrature has been used.

<u>Rectangular Elements</u>: For evaluating stiffness $[K]^{(e)}$ and consistent mass matrix $[M]^{(e)}$, one needs the second derivatives of the displacement with reference to global co-ordinates and global differential area in terms of local co-ordinates ξ, η, This is done as follows:

First, derivates of the displacement with reference to global co-ordinates are given by [5]

$$\begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} \qquad (2.26)$$

where [J] is the Jacobi matrix and is given by

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \qquad (2.27)$$

Differentiating eqn. (2.26), (first eqn. w.r.t. ξ and second eqn. w.r.t. η),

$$\begin{Bmatrix} \frac{\partial^2 N_i}{\partial \xi^2} \\ \frac{\partial^2 N_i}{\partial \eta^2} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial^2 N_i}{\partial x^2} \frac{\partial x}{\partial \xi} + \frac{\partial^2 N_i}{\partial y^2} \frac{\partial y}{\partial \xi} \\ \frac{\partial^2 N_i}{\partial x^2} \frac{\partial x}{\partial \eta} + \frac{\partial^2 N_i}{\partial y^2} \frac{\partial y}{\partial \eta} \end{Bmatrix}$$

$$= [J]\,[J] \begin{Bmatrix} \frac{\partial^2 N_i}{\partial x^2} \\ \frac{\partial^2 N_i}{\partial y^2} \end{Bmatrix} \qquad (2.28)$$

Differentiating first eqn. of eqns. (2.26) w.r.t. η , one gets

$$\frac{\partial^2 N_i}{\partial \xi \partial \eta} = \left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} + \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}\right) \frac{\partial^2 N_i}{\partial x \partial y} + \frac{\partial^2 x}{\partial \xi \partial \eta} \frac{\partial N_i}{\partial x} + \frac{\partial^2 y}{\partial \xi \partial \eta} \frac{\partial N_i}{\partial y}$$

For rectangular elements this equation simplifies to

$$\frac{\partial^2 N_i}{\partial x\,\partial y} = \frac{\partial^2 N_i}{\partial\xi\,\partial\eta} / \left(\frac{\partial x}{\partial\xi} \quad \frac{\partial y}{\partial\eta}\right) \tag{2.29}$$

For this rectangular element, using eqn. (2.17), [J] matrix given by eqn. (2.27) becomes

$$[J] = \begin{bmatrix} L_x & 0 \\ 0 & L_y \end{bmatrix} \tag{2.30}$$

It is well known that

$$\iint dx\, dy = \left| J \right| \iint d\xi \quad d\eta \tag{2.31}$$

<u>Sector Elements</u>: For these elements the above derivation is valid except that x and y should be replaced by r and θ ; and Jacobian matrix becomes

$$[J] = \begin{bmatrix} \frac{a-b}{2} & 0 \\ 0 & \frac{\theta_2-\theta_1}{2} \end{bmatrix} \tag{2.32}$$

and

$$\iint dx\, dy = \left| J \right| \iint r\, d\xi \quad d\eta \tag{2.33}$$

2.4 SOLUTION PROCEDURE

Stiffness $[K]^{(e)}$ and consistent mass $[M]^{(e)}$ matrices have been calculated using the numerical integration explained above. It may be noted that

the elements used are subparametric elements. These elemental matrices are assembled as banded matrices after applying the relevant boundary conditions. Eigenvalues are calculated from these using the NAG subroutine available on DEC 1090.

CHAPTER - III

VIBRATION ANALYSIS OF CIRCULAR AND ANNULAR PLATES WITH VARYING THICKNESS

3.1 INTRODUCTION

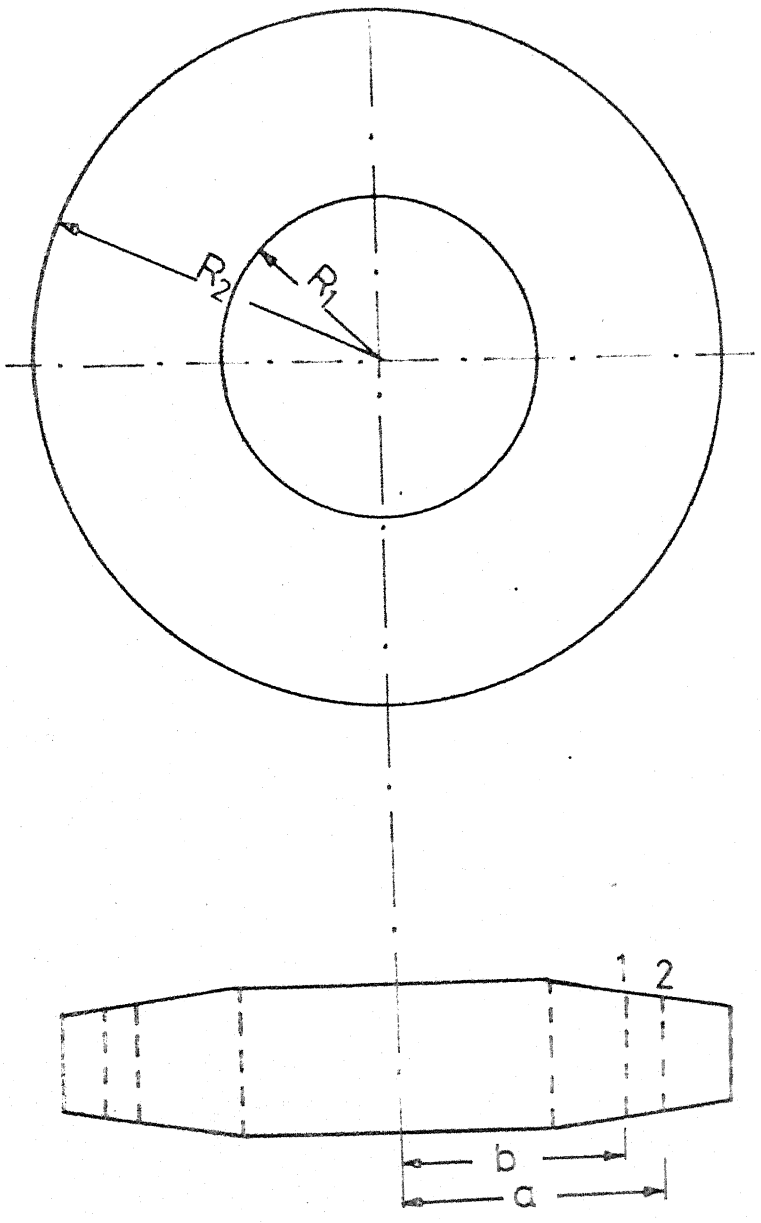
The standard finite element method described earlier is no doubt capable of dealing with any plate problem. Nevertheless the cost of solution increases rapidly with higher number of elements which are required for higher accuracy of the results. So in the case of circular or annular plates where along θ-direction there is no change either in geometry or material property semianalytical method can be used [5,9]. It reduces the two dimensional problem to a one dimensional one and hence very economical. Here we express the field variable in θ-direction in Fourier series.

3.2 FINITE ELEMENT FORMULATION

Strain energy of a finite element is given by

$$U^{(e)} = \frac{1}{2} \int_V [\varepsilon] \; [D'] \; \{\varepsilon\} \, dv \qquad (3.1)$$

Taking the displacement $W^{(e)}$ corresponding to the m^{th} harmonic over the element (Fig. 3) as



Nodal degrees of freedom are
W_i and W_{ri} ; $i=1,2$

Fig. 3 Annular element

$$W^{(e)} = (a_1 + a_2 r + a_3 r^2 + a_4 r^3) \cos m\theta$$

or

$$W^{(e)} = \lfloor S \rfloor [R] \cos m\theta \{W\}^{(ne)} \qquad (3.2)$$

where $\lfloor S \rfloor = \lfloor 1 \quad r \quad r^2 \quad r^3 \rfloor$

$$\lfloor W \rfloor^{(ne)} = \lfloor W_1 \quad W_{r_1} \quad W_2 \quad W_{r_2} \rfloor$$

and $[R]$ is given in Appendix-B.

Using eqn. (3.2), strain matrix becomes

$$\{\varepsilon\} = (-Z) \begin{bmatrix} \lfloor S,_{rr} \rfloor \cos m\theta \\ \lfloor \frac{1}{r} S,_r - \frac{m^2}{r^2} S \rfloor \cos m\theta \\ 2\lfloor -\frac{m}{r} S,_r + \frac{m}{r^2} S \rfloor \sin m\theta \end{bmatrix} [R] \{W\}^{(ne)}$$

or

$$\{\varepsilon\} = (-Z) \begin{bmatrix} 0 & 0 & 2\cos m\theta & 6r\cos m\theta \\ -\frac{m^2}{r^2}\cos m\theta & (\frac{1-m^2}{r})\cos m\theta & (2-m^2)\cos m\theta & r(3-m^2)\cos m\theta \\ \frac{m}{r}\sin m\theta & 0 & -m\sin m\theta & -2mr\sin m\theta \end{bmatrix}$$

$$\times \quad [R] \quad \{W\}^{(ne)}$$

or

$$\{\varepsilon\} = (-Z) [F'] [R] \{W\}^{(ne)} \qquad (3.3)$$

Substituting eqn. (3.3) in eqn. (3.1), and integrating w.r.t. Z **and** θ one gets

$$U^{(e)} = \frac{1}{2} \lfloor W \rfloor^{(ne)} [R]^T \int_b^a h^3 [F]^T [D] [F] \, r \, dr \, [R] \{W\}^{(ne)}$$

$$\ldots\ldots \qquad (3.4)$$

where

$$[D] = \frac{C}{12(1-\nu_r\,\nu_\theta)} \begin{bmatrix} E_r & E_r\,\nu_\theta & 0 \\ E_r\,\nu_\theta & E_\theta & 0 \\ 0 & 0 & G_{r\theta}(1-\nu_r\,\nu_\theta) \end{bmatrix} \qquad \ldots\ldots \qquad (3.5)$$

$$C = 2\pi \quad \text{when} \quad m = 0$$
$$C = \pi \quad \text{when} \quad m \geq 1 \qquad (3.6)$$

and

$$[F] = \begin{bmatrix} 0 & 0 & 2 & 6\,r \\ -\frac{m^2}{r^2} & (\frac{1-m^2}{r}) & (2-m^2) & (3-m^2)r \\ \frac{m}{r} & 0 & -m & -2mr \end{bmatrix} \qquad (3.7)$$

Eqn. (3.4) gives us the stiffness matrix $[K]^{(e)}$ as

$$[K]^{(e)} = [R]^T \int_b^a h^3\,[F]^T\,[D]\,[F]\,r\,dr\,[R]$$

or

$$= [R]^T\,[k]^{(e)}\,[R] \qquad (3.8)$$

Coefficients of $[k]^{(e)}$ are determined by closed form integration and are given in Appendix-B.

Kinetic energy of a finite element is given by

$$T^{(e)} = \frac{1}{2} \iint \rho\,h\,\dot{W}^2\,r\,dr\,d\theta \qquad (3.9)$$

Substituting eqn. (3.2) in eqn. (3.9) one gets

$$T^{(e)} = \frac{1}{2} C \lfloor \dot{W} \rfloor^{(ne)} [R]^T \int \rho h \{S\} \lfloor S \rfloor r dr [R] \{\dot{W}\}^{(ne)}$$

from which one gets the consistent mass matrix $[M]^{(e)}$ as

$$[M]^{(e)} = C [R]^T \int \rho h \{S\} \lfloor S \rfloor r \, dr [R]$$

$$= C [R]^T [m]^{(e)} [R] \qquad (3.10)$$

Coefficients of $[m]^{(e)}$ are given in appendix-B.

3.3 SOLUTION TECHNIQUE

Elemental stiffness $[K]^{(e)}$ and consistent mass $[M]^{(e)}$ matrices are assembled into global matrices. After applying the boundary conditions, the eigenvalues are obtained using the IMSL subroutine EIGZF.

CHAPTER - IV

FREE VIBRATION OF ROTATING DISCS

4.1 INTRODUCTION

The vibration of rotating discs is of much practical significance. In cases like symmetric discs used in axial flow turbomachines, formulation of the previous chapter can not be used. Now centrifugal force becomes an important factor. Its contribution to stiffness matrix must be included.To do so one needs the inplane stresses. First these inplane stresses arising due to the centrifugal force are determined. In the present work these stresses have been calculated using finite element method. The formulation is applicable to both isotropic and orthotropic discs. These inplane stresses make the disc more stiffer to flexural vibrations and hence natural frequency increases. Adding the additional stiffness matrix, obtained due to these stress, to the original stiffness matrix given in Chapter-III we get the stiffness matrix for rotating discs.

The formulation below is a generalised one which can be used for variable thickness discs (isotropic and orthotropic).

4.2 FORMULATION

The strain energy due to the inplane stresses during flexural vibration [9] is

$$U_p^{(e)} = \frac{1}{2} \int_0^{2\pi} \int_b^a [\sigma_r (\frac{\partial W}{\partial r})^2 + \frac{\sigma_\theta}{r^2} (\frac{\partial W}{\partial \theta})^2] h \, r \, dr \, d\theta \quad \ldots \quad (4.1)$$

Substituting $W^{(e)}$ from eqn. (3.2), we get stiffness matrix $[K_p]^{(e)}$ as

$$[K_p]^{(e)} = C [R]^T \int_b^a h (\sigma_r \{S,_r\} \lfloor S,_r \rfloor + \sigma_\theta \frac{m^2}{r^2} \{S\} \lfloor S \rfloor) r \, dr [R]$$

$$= C [R]^T [k_p]^{(e)} \lfloor R \rfloor \quad (4.2)$$

$$\text{where } C = 2\pi \quad \text{for } m = 0$$
$$C = \pi \quad \text{for } m \geq 1 \quad (4.3)$$

The coefficients of the matrix $[k_p]^{(e)}$ are calculated by closed form integration and are given in Appendix-C.

4.3 DETERMINATION OF INPLANE STRESSES

The inplane stresses σ_r and σ_θ needed for eqn. (4.2) can be obtained using the same annular element. The strain energy $U_i^{(e)}$ of the annular element is

$$U_i^{(e)} = \frac{1}{2}\, 2\pi \int h \lfloor \varepsilon \rfloor [D] \{\varepsilon\}\; r\, dr \qquad (4.4)$$

where

$$[D] = \frac{1}{1-\nu_r\,\nu_\theta} \begin{bmatrix} E_r & E_r\,\nu_\theta \\ & \\ E_r\,\nu_\theta & E_\theta \end{bmatrix} \text{ and } \lfloor \varepsilon \rfloor = \lfloor \frac{du}{dr} \;\; \frac{u}{r} \rfloor$$

$$\ldots \qquad (4.5)$$

Taking $u^{(e)}$ over the element as

$$u^{(e)} = (a_1 + a_2 r + a_3 r^2 + a_4 r^3) = \lfloor N_1\; N_2\; N_3\; N_4 \rfloor \begin{Bmatrix} u_1 \\ u_{r_1} \\ u_2 \\ u_{r_2} \end{Bmatrix}$$

$$= \lfloor N \rfloor \{u\}^{(ne)} \qquad \ldots\ldots \qquad (4.6)$$

where $\lfloor N \rfloor = \lfloor S \rfloor [R]$

Substituting eqn. (4.6) in eqn. (4.4) one gets

$$U_i^{(e)} = \frac{1}{2}\, 2\pi \lfloor u \rfloor^{(ne)} \int h [B]^T [D] [B]\, r\, dr \{u\}^{(ne)}$$

$$\ldots \qquad (4.7)$$

where $$[B] = \begin{Bmatrix} \lfloor N,_r \rfloor \\ \frac{\lfloor N \rfloor}{r} \end{Bmatrix} \qquad (4.8)$$

Eqn. (4.7) gives the stiffness matrix

$$[K_i]^{(e)} = 2\pi \int h [B]^T [D] [B]\, r\, dr$$

$$= [R]^T [k_i]^{(e)} [R] \qquad (4.9)$$

Coefficients of matrix $[k_i]^{(e)}$ are obtained using closed form integration and are given in Appendix-C.

The load matrix $\lfloor F \rfloor^{(ne)}$ due to body force (centrifugal force) is

$$\lfloor F \rfloor^{(ne)} = 2\pi \rho \, h \, \Omega^2 \int r^2 \; \lfloor N \rfloor \, dr \tag{4.10}$$

where ρ is the mass density, Ω is the angular velocity. These elemental matrices are assembled and displacements are obtained. Using these displacements, stresses σ_r and σ_θ are calculated using eqn. (4.5).

Using these values of σ_r and σ_θ, stiffness matrix $[K_p]^{(e)}$ for the disc is calculated. While calculating the $[K_p]^{(e)}$ stress distribution inside the element is taken linear.

i.e.

$$\sigma_r = E_1 + E_2 \, r$$

$$\sigma_\theta = D_1 + D_2 \, r$$

where

$$E_1 = \frac{\sigma r_1 \, a - \sigma r_2 \, b}{a - b} \qquad E_2 = \frac{\sigma r_2 - \sigma r_1}{a - b}$$

$$D_1 = \frac{\sigma \theta_1 \, a - \sigma \theta_2 \, b}{a - b} \qquad D_2 = \frac{\sigma \theta_2 - \sigma \theta_1}{a - b}$$

By adding $[K_p]^{(e)}$ to $[K]^{(e)}$ of plate (Chapter-III) one gets the stiffness matrix for the rotating disc. This is solved for finding out the natural frequencies of the rotating discs.

Results for isotropic plates are also calculated using the closed form solution of stresses of

the rotating disc [24], i.e.

$$\sigma_r = C_1 + \frac{C_2}{r^2} - \frac{(3 + \nu)}{3} \rho \Omega^2 r^2$$

$$\sigma_\theta = C_1 - \frac{C_2}{r^2} - \frac{(1 + 3\nu)}{8} \rho \Omega^2 r^2 \qquad (4.11)$$

For a free free disc

$$C_1 = \frac{3 + \nu}{3} \rho \Omega^2 (a^2 + b^2)$$

$$C_2 = - \frac{3 + \nu}{8} \rho \Omega^2 a^2 b^2 \qquad (4.12)$$

For a clamped free disc

$$C_2 = \left(\rho \frac{\Omega^2}{8} (R_2^2 R_1 (3 + P_r)(1-P_r) - (1-P_r^2) R_1^3) \right) / \left((1-P_r) \frac{R_1}{R_2^2} + \frac{(1+P_r)}{R_1} \right)$$

$$C_1 = \rho \Omega^2 \frac{R_2^2}{8} (3+P_r) - \frac{C_2}{R_2^2}$$

CHAPTER - V

RESULTS AND DISCUSSION

5.1 RECTANGULAR ELEMENT

The finite element program developed has been verified for convergence and accuracy. Natural frequencies of square plates with all edges free and clamped are determined and compared with results from ref. [4]. The convercence is checked by varying the number of elements taken (Table 1a). The natural frequency of a plate with a centrally located squre hole is also determined. With 12 elements the answer is found very close to the results from ref. [4]. (Table 1b). In all the cases, the natural frequency is expressed in non-dimensional form

$$\alpha = \lambda\, l^2\, (\rho\, h/D)^{1/2}$$

where $D = E\, h^3/12(1-\nu^2)$ and l is the width of the plate.

5.2 ANNULAR SECTOR ELEMENT

Here the element is used for the determination of natural frequencies of an annular sector plate with all edges clamped having a radii of 0.4 and sectorial angle of 90°. The convergence can be verified by

Comparison of the frequency parameter $\lambda\ l^2(\rho h/D)^{1/2}$ for a rectangular plate with answers from ref. [4]*

B-C	Mode	grid size 2 x 2	3 x 3	4 x 4	*
F-F	1	7.54	13.67		13.49
	2	20.23	19.82		19.79
	3	48.83	24.76		24.43
	4	56.09	35.43		35.02
C-C	1		36.75	36.05	35.99
	2		76.01	74.03	73.41
	3		76.33	74.03	73.41
	4		113.61	109.43	108.30

Table 1a.

Comparison of frequency parameter for a clamped square plate with a square hole of side length l/2 located at the centre with answers from ref. [4]*

B-C	grid size	F.E.M.	*
C-C	4 x 4	66.09	62.40

Table 1b.

Comparison of the frequency parameter $\lambda\, R_2^2(\rho h/D)^{1/2}$ of a clamped annular sector plate with answers from ref.[26]*

Mode	grid size 4 x 4	5 x 5	*
1	53.68	53.41	52.70
2	91.90	91.60	88.80
3	123.10	122.07	125.10
4	153.70	152.76	140.00
5	162.30	157.12	170.40

Table 1c.

Comparison of the frequency parameter $\lambda\, R_2^2(\rho h/D)^{1/2}$ for F-F annular plate with answers from ref. [4]*

Mode	R_1/R_2	Grid 7 x 2	*
1	0.5	4.51	4.28
2		12.10	9.32

Table 1d.

comparing the results tabulated with 4 x 4 grid and 5 x 5 grid (Table 1c). The answers from F.E.M. are expected to be higher than the exact values. Here it is violated for some of the modes. This may be due to the unequal size of the elements used. By increasing the number of elements, required accuracy in answers can be obtained. Natural frequencies of a free-free annular plate is tabulated in Table 1d.

5.3 ANNULAR ELEMENT

5.3a Isotropic Plates

Because of the non-availability of the results for orthotropic plates for all the boundary conditions, first the results obtained by the semianalytical method for isotropic plates are compared with exact values wherever available as a verification of the program. In other cases only the convergence of the results is verified.

The vibration analysis of a solid circular plate can also be carried out using this method making an assumption that a small hole ($R_1/R_2 = 0.001$) exists at the centre. It is found that the natural frequency of an annular plate with radii ratio 0.01 itself is very close to that of a solid plate. As the ratio decreases further the frequency parameter remains almost stationary. These results are tabulated in Table 2.

Frequency parameter $\lambda\ R_2^2(\rho h/D)^{1/2}$ for a solid circular isotropic plate for various boundary conditions with answers from ref. [4]*

Ne = 4

B-C	n	m	R_1/R_2 0.01	R_1/R_2 0.001	*
Free	1	0	9.00	9.00	9.03
	1	1	20.43	20.43	20.52
	0	2	5.35	5.35	5.25
	0	3	12.43	12.44	12.23
Simple	0	0	4.93	4.93	4.97
	0	1	13.90	13.90	13.94
	0	2	25.62	25.62	25.65
	1	1	29.74	29.74	29.76
Clamped		0	10.21	10.21	10.22
			21.26	21.26	21.26
	0	2	34.91	34.91	34.88
	1	1	40.24	40.24	39.77

Table 2

In case of annular plates the variation of frequency parameter with radii ratio is shown in Figs. 4 and 5. In case of a free-free plate the frequency decreases with increase in radii ratio. In all other cases considered it goes on increasing with the radii ratio.

5.3b Orthotropic Plates

The orthotropic plates are studied with three cases of boundary conditions namely simple-clamped, clamped-clamped and free-clamped. The above combination is chosen because of the availability of the results for the above mentioned boundary conditions [20]. The results are tabulated in Table 3.

Orthotropic plates having different D_θ/D_R ratios are studied keeping D_r constant. From Figs. 6, 7 and 8 it is clear that the frequency parameter $\alpha = \lambda\ (R_2^2)$ $(\rho h/D_R)^{1/2}$ increases with D_θ/D_R. But variation is very small. This shows that the material property D_r has the maximum influence on α compared to D_θ and $D_{r\theta}$. It can also be observed from Figs. 6, 7 and 8 that α increases with D_θ/D_R ratio linearly.

The natural frequency of annular plates (both isotropic and orthotropic) with varying thickness is also determined using the same element. Here a linear

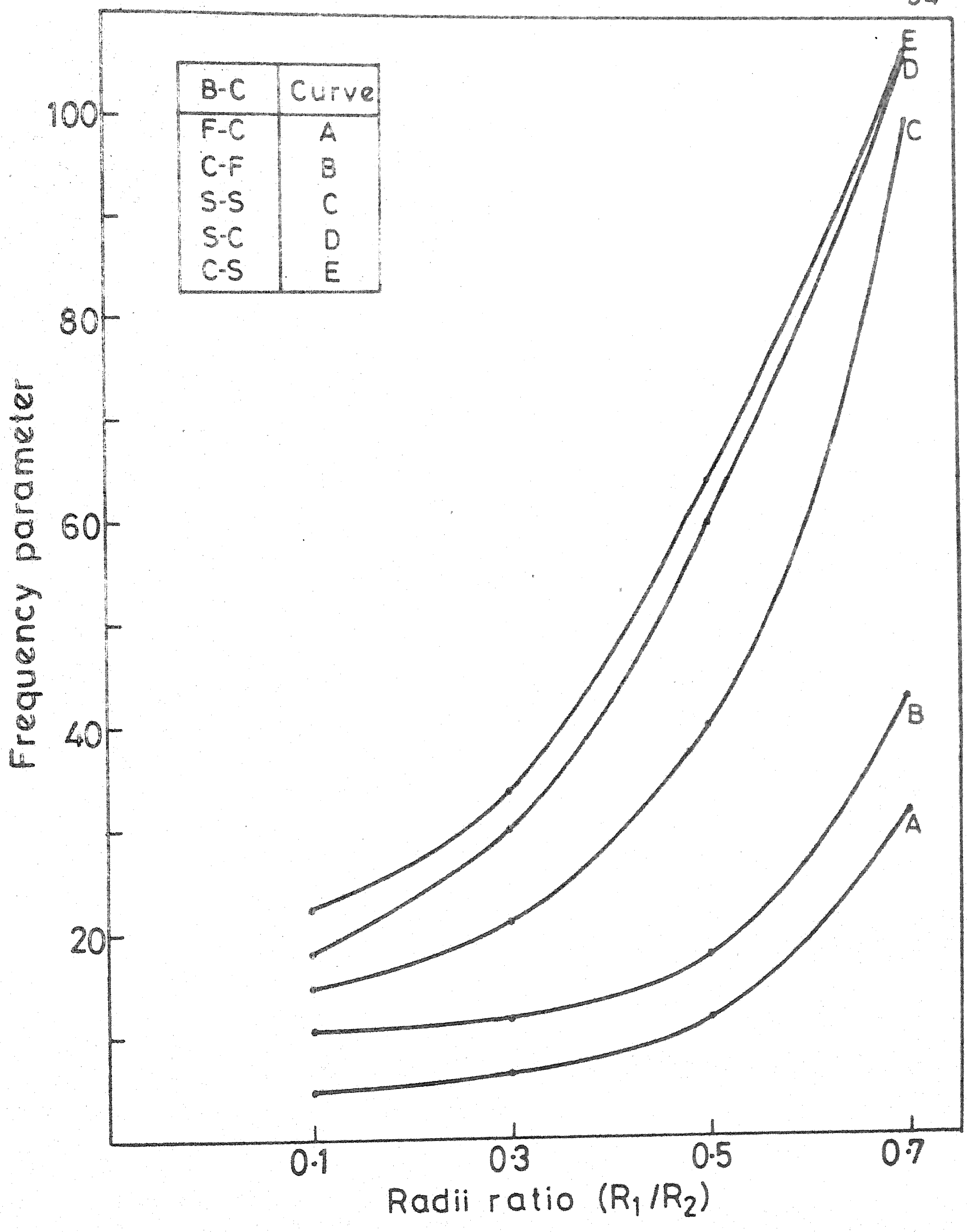
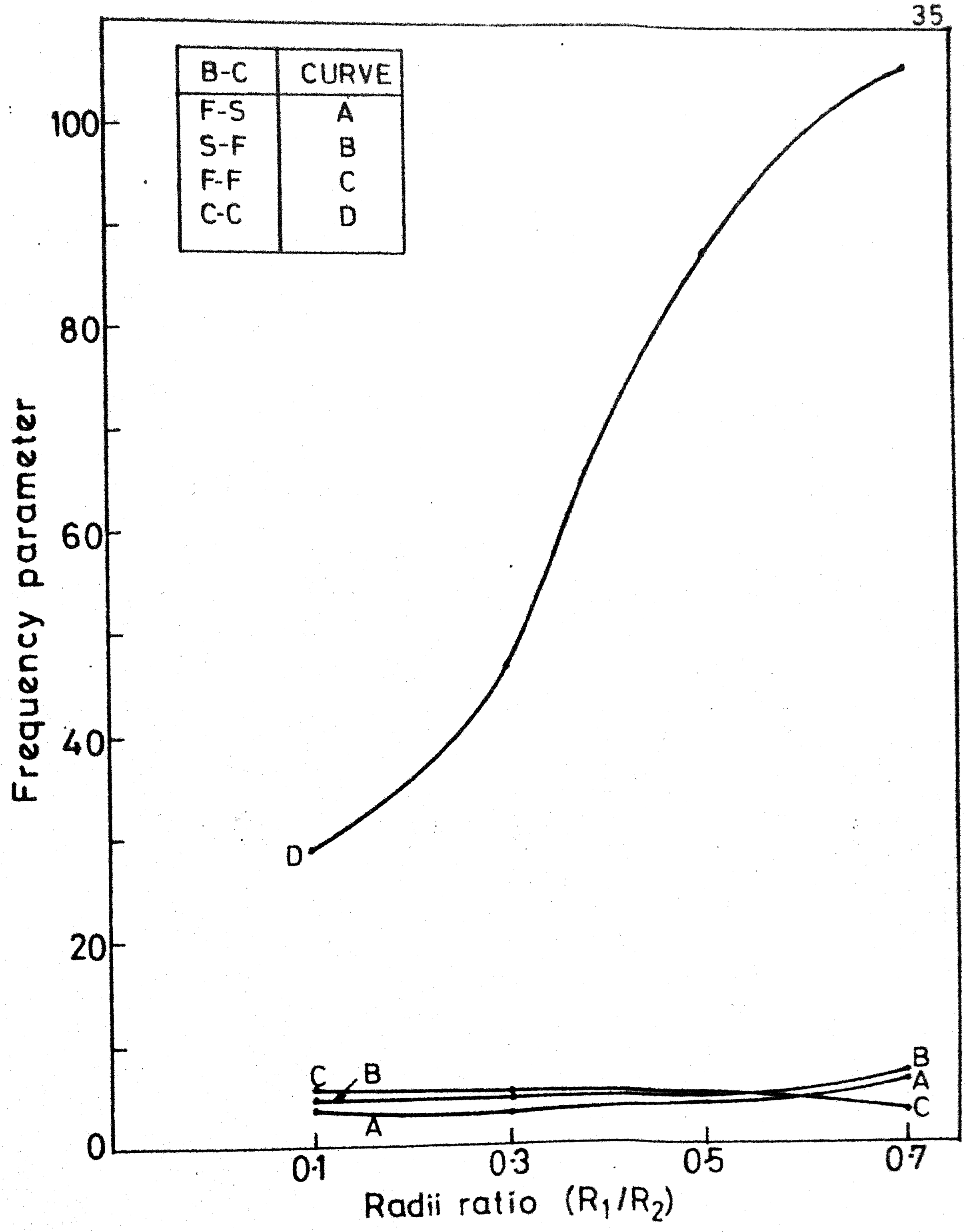


Fig. 4 Variation of the frequency parameter with radii ratio for an isotropic annular plate


B-C
CURVE
F-S
A
S-F
B
F-F
C
C-C
D
Frequency parameter
100
80
60
40
20
0
D
C
B
A
B
A
C
0·1
0·3
0·5
0·7
Radii ratio (R_1/R_2)

Comparison of the frequency parameter $\lambda\, R_2^2(\rho h/D_r)$ with the answers from ref. [20]*

$D_\theta/D_r = 2.0 \quad D_{r\theta}/D_r = 0.35 \quad \nu_\theta = 0.3$

B-C	n	m	Ne	R_1/R_2 = 0.1	0.3	0.5
F-F	0	2	1	6.516	6.220	5.570
			2	6.502	6.210	5.567
			4	6.497	6.205	5.666
	*					
	2	0	1	11.599	11.383	13.453
			2	11.583	11.337	13.444
			4	11.513	11.303	13.436
	*			11.470		13.503
	1	3	1	15.365	15.667	15.036
			2	15.675	15.620	15.025
			4	15.672	16.610	15.021
	*			15.760		15.600
S-F	0	0	1	6.261	6.516	7.294
			2	6.213	6.459	7.269
			4	6.203	6.449	7.226
	*			6.250		7.300
	0	1	1	15.433	14.774	13.790
			2	14.923	14.423	13.668
			4	14.900	14.330	13.654
	*					
	0	2	1	31.694	23.927	26.602
			2	28.803	27.740	26.123
			4	28.650	27.699	26.083
	*			28.500		26.210
C-F	0	0	1	11.826	13.252	19.156
			2	11.700	13.136	19.089
			4	11.663	13.107	19.075
	*			11.690		19.075
	0	1	1	24.142	22.3[illegible]0	24.205
			2	22.436	21.656	23.954
			4	22.353	21.537	23.920
	*					
	0	2	1	53.137	40.619	37.205
			2	38.613	37.103	36.159
			4	38.394	36.954	36.045
	*			38.300		36.090

Table 3.

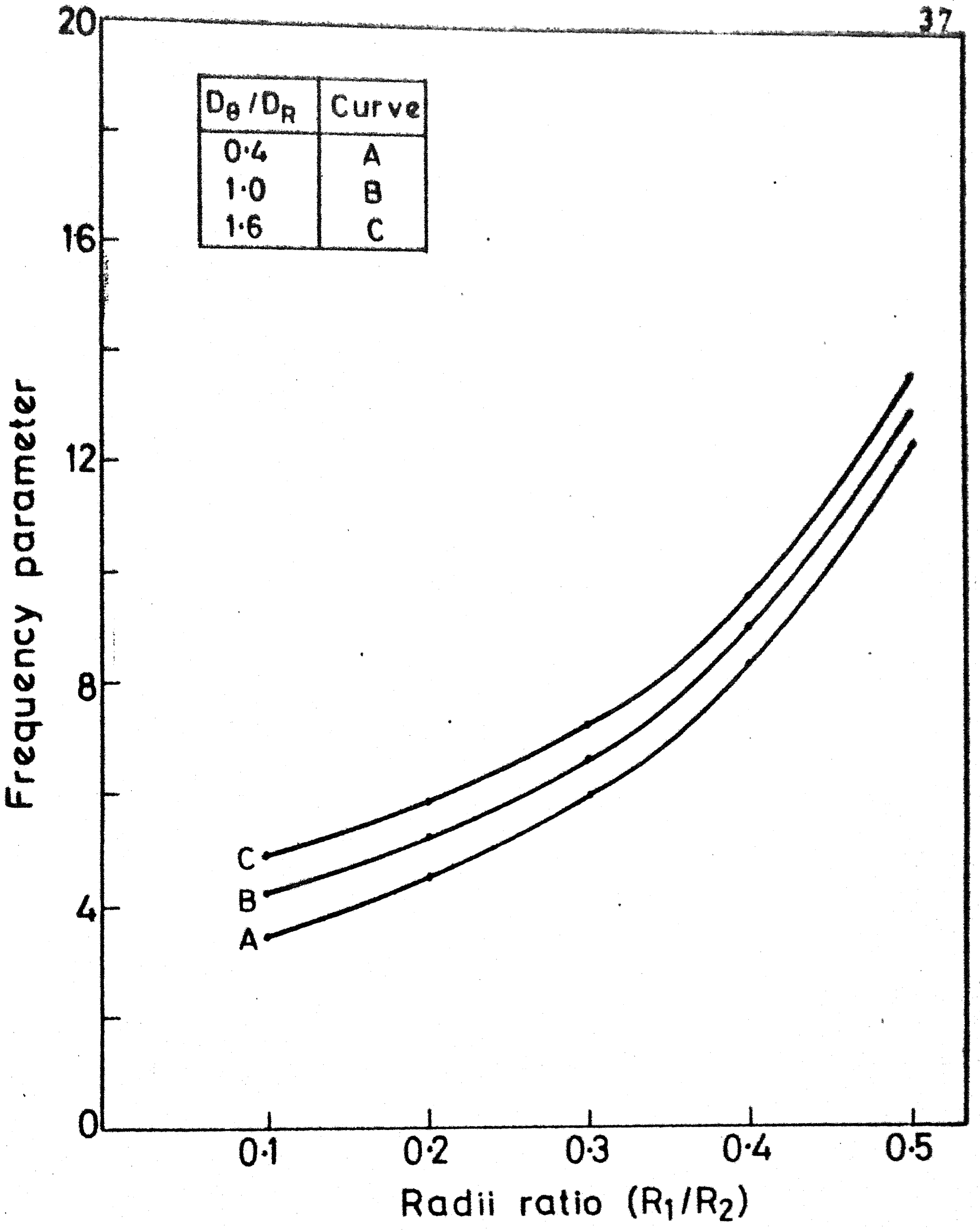


Fig.6 Variation of the frequency parameter with radii ratio for different D_θ/D_R ratios for a F-C plate

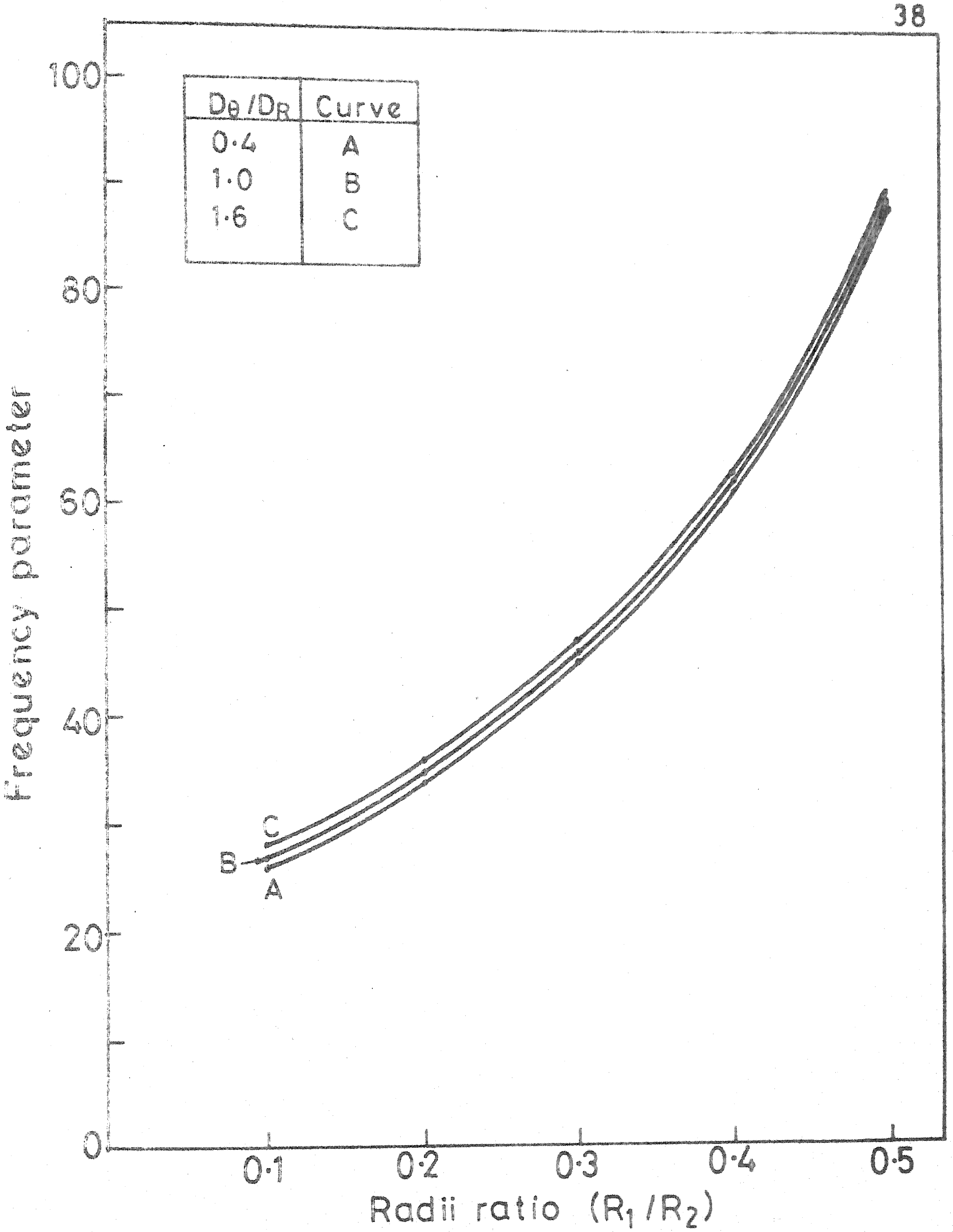


Fig.7 Variation of the frequency parameter with radii ratio for different D_θ/D_R ratio for a C-C plate

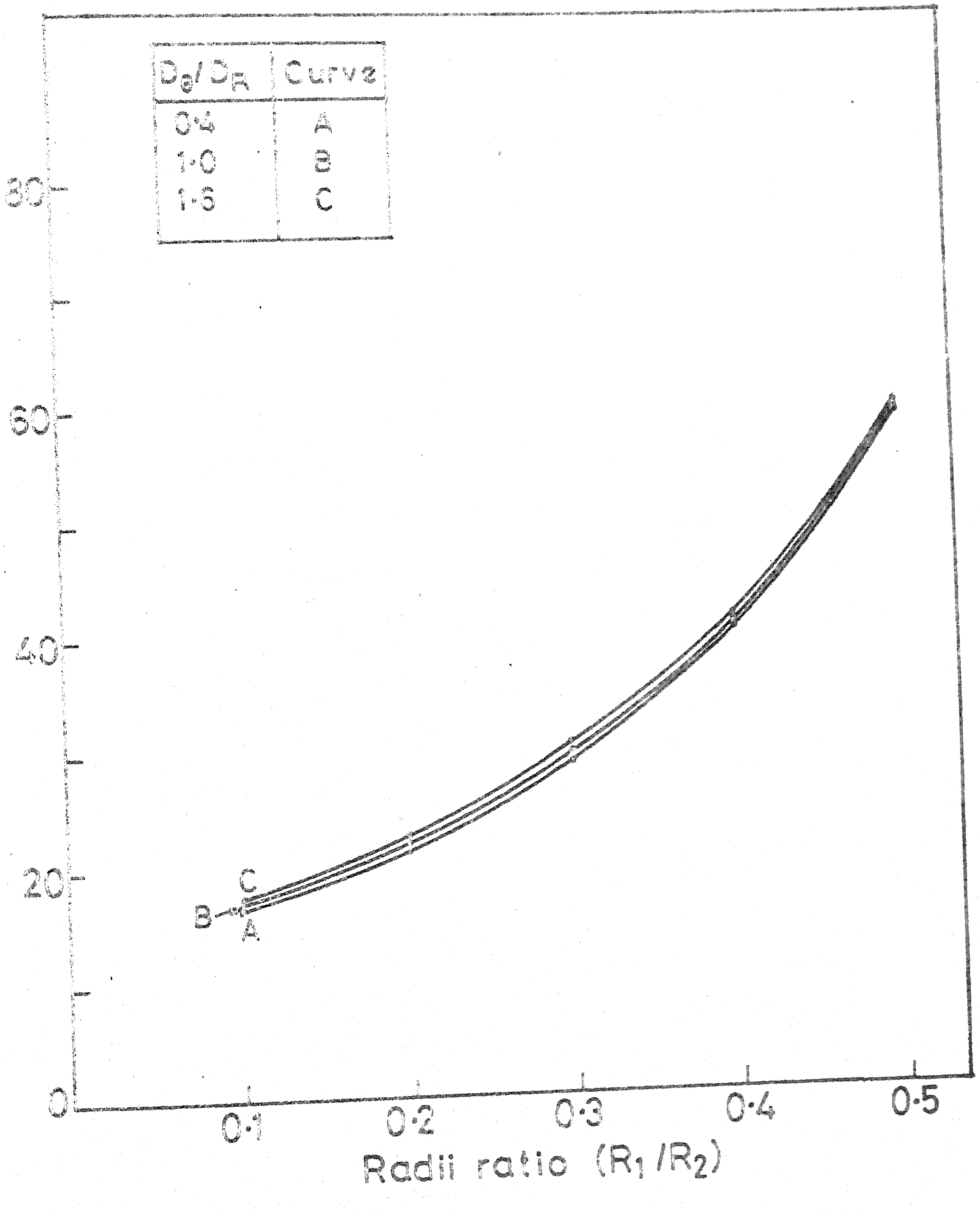


Fig.8 Variation of the frequency parameter with radii ratio for different (D_θ/D_r) ratios for a S-C plate

variation in thickness along the radial direction is considered. The thickness at outer radius R_2 is kept constant (h_2=0.1 cm). The thickness at the inner radius R_1 (h_1) is varied from 0.08 to 0.12 cm in steps of 0.01 cm. In this case frequency parameter α is expressed as $\alpha = \lambda R_2^2 \ (12 \rho (1-\nu_r \ \nu_\theta)/(h_2^2 \ E_r))^{1/2}$. Figs. 9 and 10 show the variation of α with radii ratio for different thickness ratios. From Fig. 11 it can be noticed that variation with thickness ratio is almost linear when thickness variation is small.

When D_θ/D_R and h_1/h_2 are = 1 it is the case of isotropic uniform thickness plates. Hence in this case values are compared with exact values (Table 5). It is a verification of the program for its dependibility. In Table 4 and 6 convergence of the results for varying thickness plates is verified.

Table 7, 8 and 9 give the values of frequency parameter for different D_θ/D_R and h_1/h_2 ratios for simple-clamped, clamped-clamped and free-clamped plates respectively.

5.3c Rotating Isotropic Discs

The dynamic behaviour of rotating discs are studied for the two most commonly used boundary conditions. They are inner edge clamped or free and outer

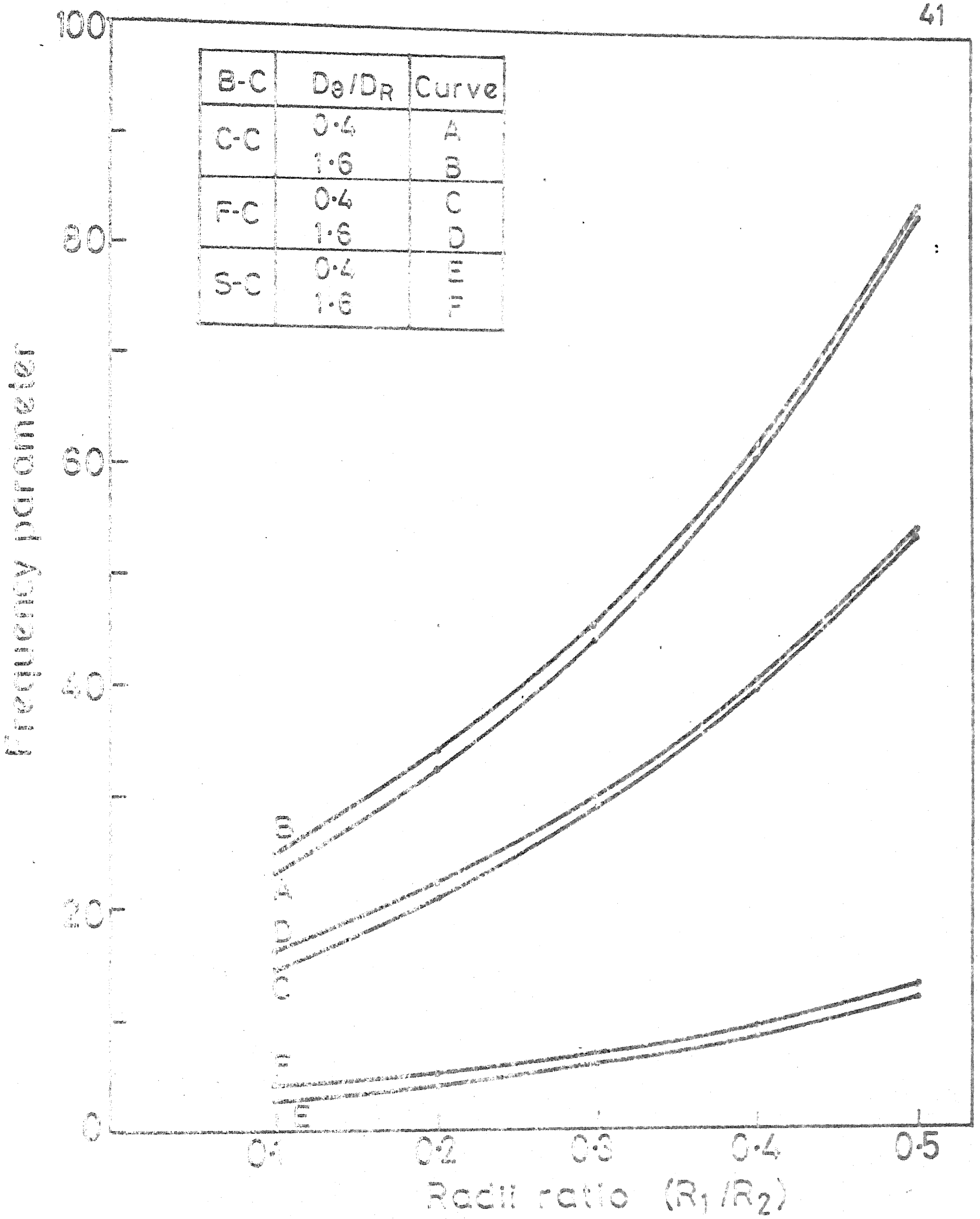


Fig.9 Variation of the frequency parameter with radii ratio for a plate with thickness ratio 0·8

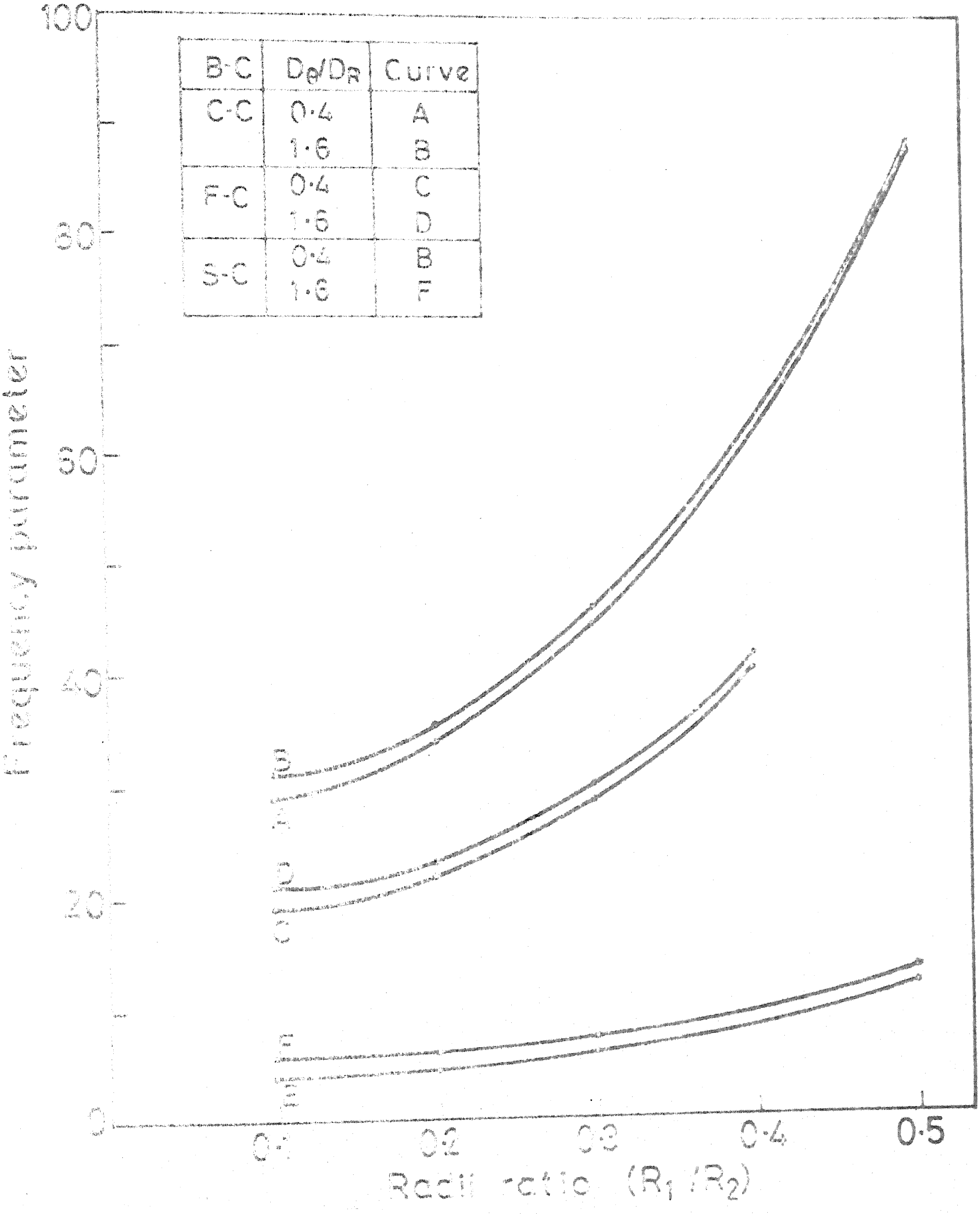


Fig.10 Variation of frequency parameter with radii ratio for a plate with thickness ratio 1·2

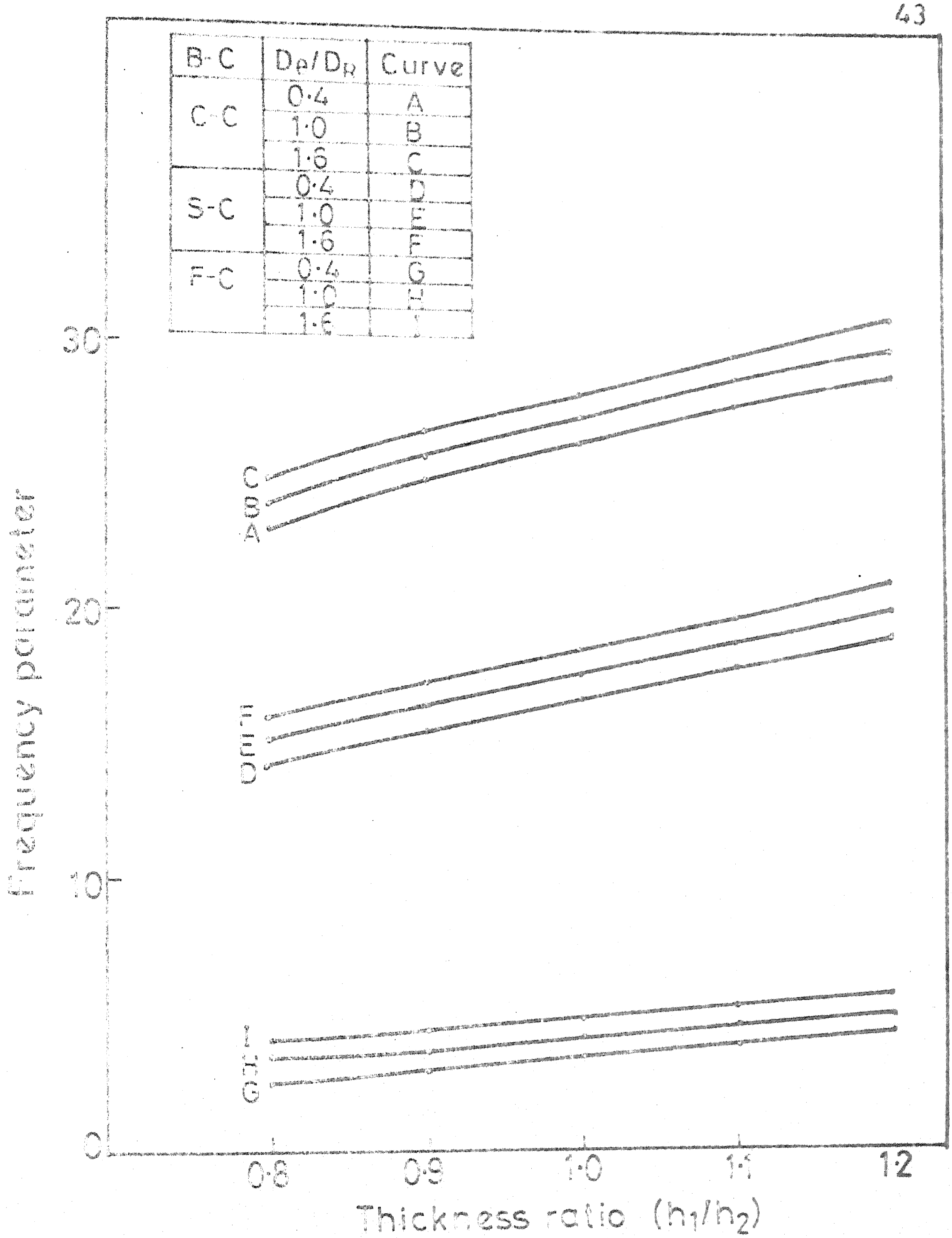


Fig.11 Variation of frequency parameter with thickness ratio (h_1/h_2) for different (D_θ/D_R) ratios

Verification of the convergence of frequency parameter $\lambda R_2^2(12\rho\,(1-\nu_r\,\nu_\theta)/(E_r\,h_1^2))^{1/2}$ for orthotropic plates with linearly varying thickness

$D_\theta/D_r = 0.4$ $D_{r\theta}/D_r = 0.35$ $\nu_\theta = 0.3$

$h_1/h_2 = 0.8$

B-C	n	m	Ne	R_1/R_2 = 0.1	0.2	0.3	0.4	0.5
F-C	0	0	8	3.74	6.22	8.39	11.77	17.43
			10	3.76	6.22	8.39	11.77	17.43
	0	1	8	4.45	6.40	8.79	12.34	18.14
			10	4.47	6.40	8.79	22.34	18.14
	0	2	8	6.66	8.34	10.77	14.46	20.46
			10	6.66	8.34	10.77	14.46	20.46
	0	3	8	12.88	13.37	15.56	18.89	24.77
			10	12.87	13.72	15.56	18.89	24.77
C-C	0	0	8	32.95	47.49	62.71	86.05	124.25
			10	33.01	47.51	62.71	86.21	124.18
	0	1	8	39.13	49.80	64.61	87.87	125.64
			10	39.21	49.78	64.61	87.83	125.57
	0	2	8	48.49	56.84	70.42	92.57	129.83
			10	48.48	56.82	70.42	92.74	129.75
	0	3	8	63.41	68.89	80.29	101.05	136.87
			10	63.43	68.86	80.29	100.86	136.79
						h_1/h_2 =-0.9		
F-C	0	0	8	3.56	5.22	7.03	9.86	14.60
			10	3.56	5.22	7.03	9.86	14.60
	0	1	8	3.78	5.36	7.36	10.34	15.20
			10	3.79	5.36	7.36	10.34	15.20
	0	2	8	5.59	6.99	9.03	12.12	17.15
			10	5.58	6.98	9.02	12.12	17.15
	0	3	8	10.79	11.50	13.04	15.83	20.76
			10	10.79	11.50	13.04	15.83	20.75
C-C	0	0	8	29.25	39.79	52.57	72.08	104.13
			10	29.27	39.80	52.58	72.21	104.44
	0	1	8	33.00	41.71	54.16	73.44	105.29
			10	33.02	41.70	54.17	73.57	105.61
	0	2	8	40.66	47.66	59.04	74.53	108.82
			10	40.64	47.61	59.05	77.68	109.14
	0	3	8	53.17	57.77	67.31	84.45	114.74
			10	53.15	57.71	67.34	84.63	115.07

Table 4.

Comparison of the frequency parameter $\lambda R_2^2(12\rho(1-\nu_r \nu_\theta)/(E_r h_1^2))^{1/2}$ of orthotropic plate answers from ref.[20]*

$D_\theta/D_r = 10$ $D_{r\theta}/D_r = 0.35$ $\nu_\theta = 0.3$ Ne = 8

B-C	n	m	R_1/R_2				
			0.1	0.2	0.3	0.4	0.5
		0	27.31	34.61	45.37	62.00	89.16
		*	27.30	-	45.20	-	89.20
		1	28.95	36.11	46.67	63.13	90.14
C-C	0	*	28.40	-	45.60	-	90.20
		2	36.65	41.83	51.18	66.82	93.22
		*	36.70	-	51.00	-	93.30
		3	51.25	53.38	60.09	73.80	98.80
		*	51.20	-	60.00	-	99.00
		0	17.80	22.71	29.98	41.26	59.82
		*	17.80	-	29.90	-	59.80
		1	19.42	24.27	31.40	42.56	60.99
		*	19.00	-	31.40	-	61.00
S-C	0	2	26.74	30.09	36.25	56.72	64.64
		*	26.00	-	36.20	-	64.60
		3	40.08	41.32	45.47	54.38	71.10
		*	40.00	-	45.40	-	71.00
		0	4.23	5.18	6.66	9.02	13.02
		*	4.23	-	6.66	-	13.00
		1	3.48	4.81	6.55	9.11	13.29
F-C	0	*	3.14	-	6.63	-	13.30
		2	5.62	6.45	7.95	10.46	14.70
		*	5.62	-	7.95	-	14.70
		3	12.45	12.61	13.27	14.96	18.56
		*	12.40	-	13.30	-	18.50

Table 5

Verification of the convergence of frequency parameter $\lambda R_2^2(12\rho(1-\nu_r\nu_\theta)/(E_r h_1^2))^{1/2}$ for orthotropic plates with linearly varying thickness

$D_\theta/D_r = 0.4 \quad D_{r\theta}/D_r = 0.35 \quad \nu_\theta = 0.3$

$h_1/h_2 = 1.1$

B-C	n	m	Ne	R_1/R_2				
				0.1	0.2	0.3	0.4	0.5
C-F	0	0	8	3.28	3.86	5.20	7.30	10.81
			10	3.27	3.86	5.20	7.29	10.81
		1	8	2.88	3.97	5.45	7.65	11.25
			10	2.87	3.96	5.45	7.65	11.25
		2	8	4.14	5.17	6.68	8.97	12.69
			10	4.14	5.17	6.68	8.96	12.69
		3	8	7.99	8.51	9.65	11.72	15.36
			10	7.98	8.51	9.65	11.71	15.36
C-C	0	0	8	24.03	29.45	38.90	53.34	77.07
			10	23.98	29.46	28.87	53.48	78.12
		1	8	24.72	30.88	40.08	54.36	77.94
			10	24.67	30.88	40.05	54.49	79.00
		2	8	30.13	35.27	43.69	57.38	80.53
			10	30.10	35.27	43.64	57.54	81.67
		3	8	39.35	42.74	49.81	62.50	84.89
			10	39.34	42.72	49.75	62.69	86.16
				$h_1/h_2 = 1.2$				
C-F		0	8	3.18	3.39	4.57	6.40	9.48
			10	3.17	3.39	4.57	6.40	9.48
		1	8	2.56	3.48	4.78	6.71	9.87
			10	2.55	3.48	4.78	6.71	9.87
		2	8	3.64	4.55	5.86	7.87	11.13
			10	3.63	4.53	5.86	7.87	11.13
		3	8	7.01	7.47	8.47	10.28	13.48
			10	7.01	7.47	8.47	10.28	13.48
C-C		0	8	22.12	35.86	34.15	46.81	67.64
			10	22.05	25.85	34.13	46.92	67.58
		1	8	21.83	27.09	35.18	47.69	68.40
			10	21.75	27.09	35.17	47.80	68.34
		2	8	26.47	30.94	38.34	50.35	70.68
			10	26.43	30.95	38.32	50.47	70.62
		3	8	34.53	37.51	43.71	54.85	74.51
			10	34.54	37.48	43.70	54.99	74.45

Table 6.

Frequency parameter $\lambda R_2^2 (12\rho(1-\nu_r \nu_\theta)/(E_r h_2^2))^{1/2}$ of a S-C plate

Ne = 8 $D_{r\theta}/D_r = 0.35$ $\nu_\theta = 0.3$

D_θ/D_r	h_1/h_2	R_1/R_2				
		0.1	0.2	0.3	0.4	0.5
0.4	0.8	14.47	21.66	29.39	40.73	59.32
	0.9	15.67	21.85	29.43	40.76	59.53
	1.0	16.86	22.04	29.45	40.82	59.45
	1.1	18.04	22.24	29.50	40.71	59.71
	1.2	19.21	22.43	29.50	40.75	59.14
0.7	0.8	14.90	21.99	29.65	40.95	59.50
	0.9	16.13	22.18	39.70	40.98	59.71
	1.0	17.34	22.38	29.72	41.04	59.63
	1.1	18.55	22.58	29.77	41.02	59.90
	1.2	19.75	22.77	29.77	40.97	59.32
1.0	0.8	15.31	22.32	29.91	41.46	59.69
	0.9	16.56	22.51	29.96	41.20	59.90
	1.0	17.80	22.71	29.98	41.26	59.82
	1.1	19.04	22.92	30.03	41.24	60.09
	1.2	20.26	23.11	30.03	41.19	59.51
1.3	0.8	15.71	22.64	30.17	41.38	59.88
	0.9	16.98	22.84	30.22	41.41	60.09
	1.0	18.25	23.04	30.24	41.48	60.01
	1.1	19.51	23.25	30.29	41.46	60.28
	1.2	20.76	23.44	30.29	41.41	59.70
1.6	0.8	16.09	22.96	30.43	41.60	60.07
	0.9	17.39	23.16	30.48	41.63	60.28
	1.0	18.68	23.36	30.50	41.70	60.20
	1.1	19.97	23.57	30.55	41.68	60.47
	1.2	21.25	23.76	30.55	41.62	59.88

Table 7

Frequency parameter $\lambda R_2^2 (12\rho(1-\nu_r\nu_\theta)/(E_r h_2^2))^{1/2}$ of a C-C plate

$Ne = 8 \quad D_{r\theta}/D_r = 0.35 \quad \nu_\theta = 0.3$

D_θ/D_r	h_1/h_2	R_1/R_2				
		0.1	0.3	0.3	0.4	0.5
0.4	0.8	23.58	33.53	44.82	61.47	88.54
	0.9	24.98	33.76	44.88	61.51	88.62
	1.0	26.36	33.98	44.90	61.62	88.35
	1.1	27.72	34.21	44.93	61.29	89.29
	1.2	29.08	34.43	44.95	61.95	88.04
0.7	0.8	23.98	33.85	45.06	61.66	88.69
	0.9	25.42	34.07	45.12	61.70	88.77
	1.0	26.84	34.30	45.14	61.81	89.01
	1.1	28.24	34.53	65.17	61.83	89.45
	1.2	29.63	34.75	45.18	61.73	88.19
1.0	0.8	24.40	34.15	45.30	61.85	88.85
	0.9	25.86	34.38	45.36	61.89	88.93
	1.0	27.31	34.61	45.37	62.00	89.16
	1.1	28.74	34.85	45.40	62 02	89.60
	1.2	30.16	35.07	45.42	61.92	88.34
1.3	0.8	24.79	34.46	45.53	62.04	89.00
	0.9	26.28	34.69	45.54	62.08	89.08
	1.0	27.76	34.92	45.61	62.19	89.32
	1.1	29.23	35.16	45.64	62.21	89.76
	1.2	30.68	35.39	45.66	62.11	89.49
1.6	0.8	25.17	34.76	45.76	62.23	89.16
	0.9	26.70	34.99	45.82	62.27	89.24
	1.0	28.21	35.23	45.84	62.38	89.48
	1.1	29.70	35.47	45.87	62.39	89.92
	1.2	31.19	35.70	45.89	62.29	89.65

Table 8

Frequency parameter $\lambda R_2^2 (12 \rho (1-\nu_r \nu_\theta)/(E_r h_2^2))^{1/2}$ of a F-C plate

$Ne = 8 \quad D_{r\theta}/D_r = 0.35 \quad \nu_\theta = 0.3$

D_θ/D_r	h_1/h_2	R_1/R_2				
		0.1	0.2	0.3	0.4	0.5
0.4	0.8	2.68	4.33	5.98	8.42	12.47
	0.9	3.03	4.39	5.99	8.42	12.47
	1.0	3.40	4.45	6.00	8.42	12.47
	1.1	3.79	4.52	6.02	8.42	12.46
	1.2	4.18	4.58	6.03	8.42	12.47
0.7	0.8	3.08	4.70	6.322	8.72	12.74
	0.9	3.45	4.77	6.33	8.72	12.75
	1.0	3.84	4.83	6.34	8.72	12.75
	1.1	4.24	4.89	6.35	8.72	12.75
	1.2	4.66	4.96	6.36	8.73	12.75
1.0	0.8	3.43	5.04	6.63	9.01	13.02
	0.9	3.83	5.11	6.64	9.02	13.02
	1.0	4.23	5.18	6.66	9.02	13.02
	1.1	4.66	5.24	6.67	9.02	13.02
	1.2	5.09	5.31	6.68	9.02	13.02
1.3	0.8	3.75	5.37	6.94	9.30	13.28
	0.9	4.16	5.43	6.95	9.30	13.29
	1.0	4.59	5.50	6.96	9.30	13.29
	1.1	5.04	5.57	6.97	9.30	13.28
	1.2	5.49	5.64	6.98	9.30	13.29
1.6	0.8	4.04	5.67	7.22	9.57	13.55
	0.9	4.48	5.74	7.24	9.57	13.55
	1.0	4.93	5.81	7.25	9.57	13.55
	1.1	5.39	5.88	7.26	9.57	13.55
	1.2	5.86	5.95	7.27	9.58	13.55

Table 9

edge free.

Due to the extra stiffness provided by the inplane stresses the value of α increases as expected. The variation of frequency parameter with different radii ratio is shown in Figs. 12 and 13 for different speeds for the two boundary conditions, free-free and free-clamped respectively. The fundamental frequency parameter is considered in both the cases.

As in the case of free plates, the frequency decreases with radii ratio. The influence of rotation is more in the case of free-free disc than free-clamped disc. In Table 10 frequency parameters for free-clamped and free-free discs for various speeds and radii ratio are tabulated. In Table 11 convergence is verified by comparing the values obtained by 4 and 6 elements. The frequency parameters obtained using the inplane stresses from F.E.M. and exact solutions are compared in Table 12.

5.3d Rotating Orthotropic Discs

The variation of frequency parameter with speed is similar to that of isotropic plate, but for the change in magnitude. As in the case of isotropic plates it can be observed from Figs. 14, 15 and 16 that variation is more in the case of free-free discs than free-clamped discs. The frequency parameters for various D_θ/D_r ratios and radii ratios are given in Table 13.

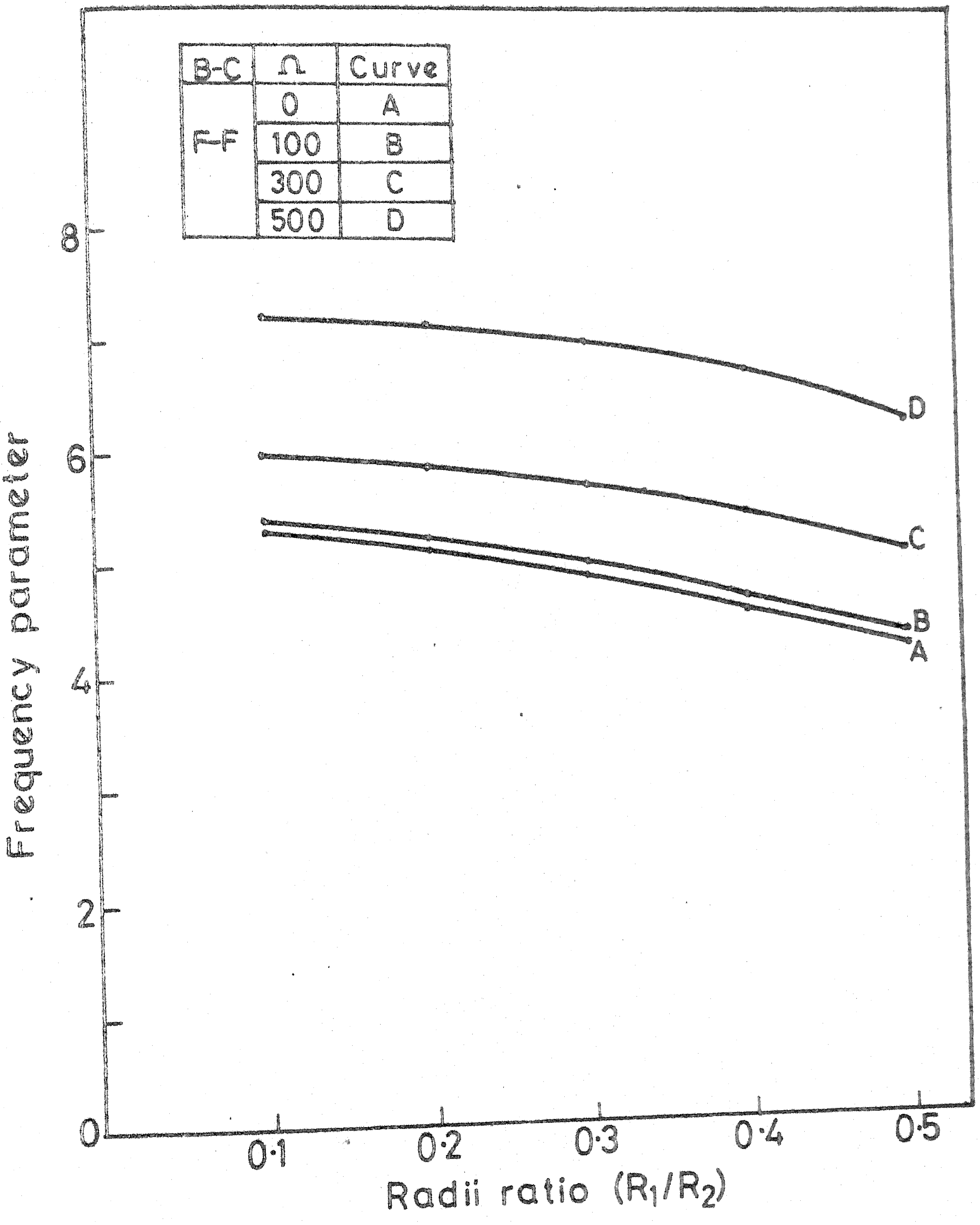


Fig.12 Variation of frequency parameter with radii ratio for different speeds

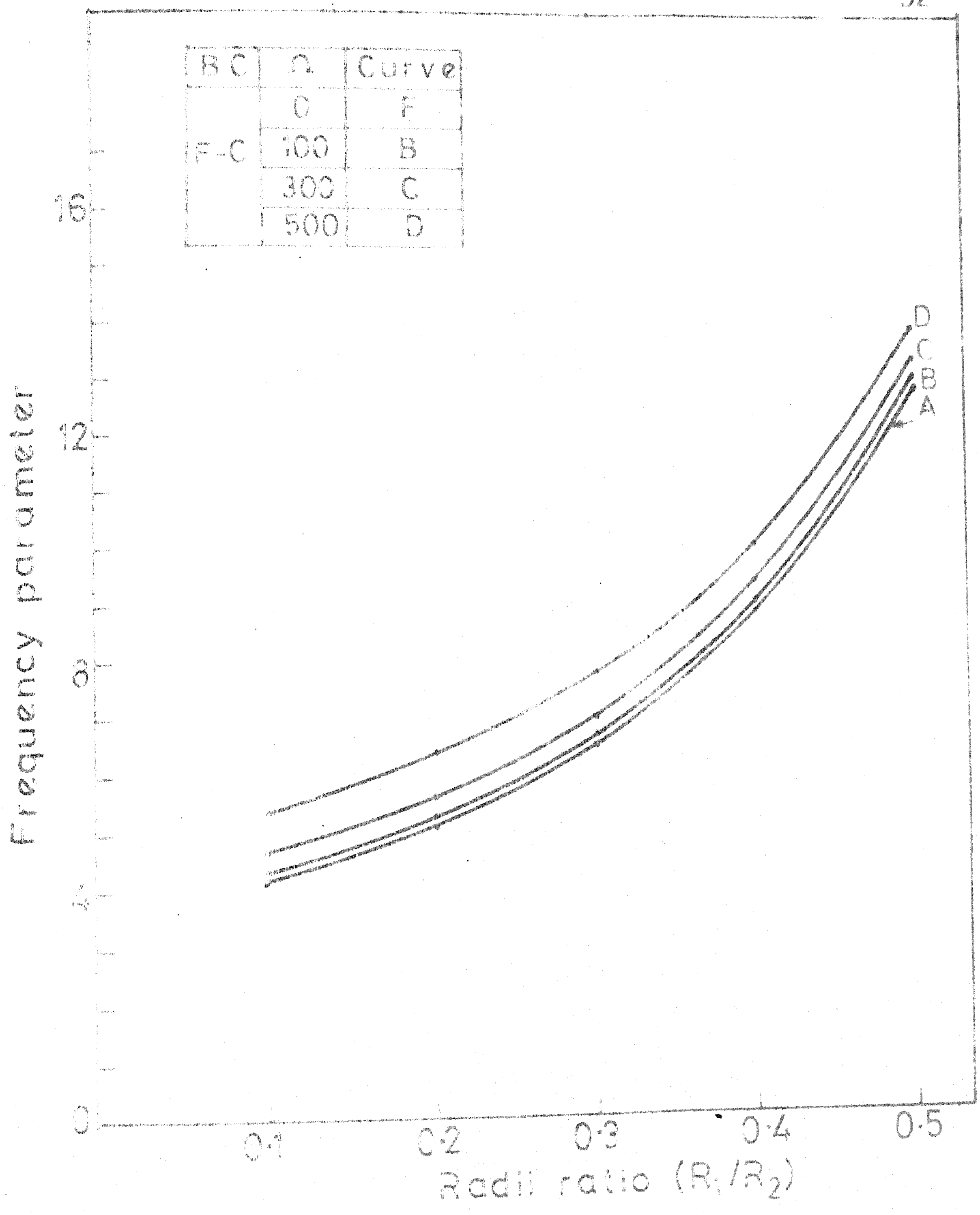


Fig.13 Variation of frequency parameter with radii ratio for different speeds

Variation of the frequency parameter $\lambda R_2^2(\rho h/D)^{1/2}$ with angular velocity Ω for different radii ratio (R_1/R_2) (inplane stresses from exact solution)

Ne = 5

B-C	Ω	R_1/R_2				
		0.1	0.2	0.3	0.4	0.5
	100	4.29	5.23	6.71	9.06	13.06
	200	4.44	5.38	6.85	9.19	13.16
F-C	300	4.67	5.62	7.08	9.40	13.33
m=0, n=0	400	4.97	5.94	7.39	9.68	13.57
	500	5.31	6.31	7.76	10.02	13.87
	100	5.39	5.24	5.01	4.71	4.38
	200	5.66	5.52	5.30	5.03	4.71
	300	6.07	5.95	5.76	5.50	5.17
F-F	400	6.61	6.51	6.34	6.07	5.71
m=2, n=0	500	7.24	7.16	6.99	6.69	6.26

Table 10

Verification of convergence of the frequency parameter $\lambda R_2^2 (\rho h/D)^{1/2}$ for the rotating disc (Inplane stresses from exact solution)

$\nu_r = 0.3 \quad R_1/R_2 = 0.5$

B-C	R.P.M.	m = 2, n = 0 Ne = 4	Ne = 6
F-F	100	4.38	4.38
	200	4.71	4.71
	300	5.17	5.17
	400	5.70	5.71
	500	6.26	6.27
F-C	100	14.74	14.74
	200	14.86	14.85
	300	15.04	15.04
	400	15.30	15.31
	500	15.63	15.63

Table 11

Comparison of the frequency parameter $\lambda R_2^2 (\rho h/D_r)^{1/2}$ determined using inplane stresses from exact solutions and finite element methods.

$\Omega = 300$, $\nu = 0.3$, $Ne = 14$

B-C	R_1/R_2	F.E.M.	Exact
	0.1	4.74	4.67
F-C	0.3	7.14	7.08
m=0, n=0	0.5	13.37	13.33
	0.1	6.30	6.07
F-F	0.3	5.94	5.76
m=2 n=0	0.5	5.25	5.19

Table 12

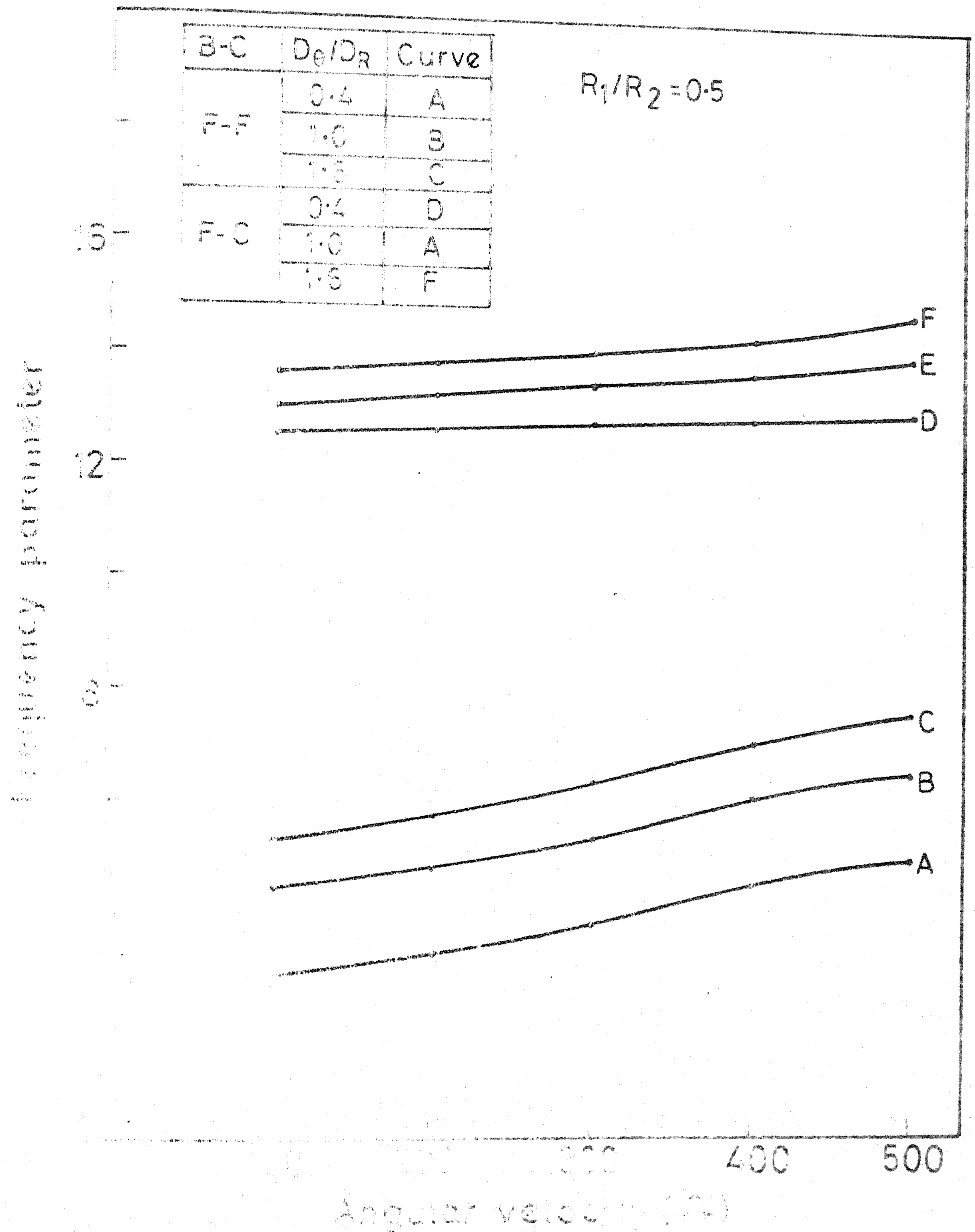


Fig.14 Variation of frequency parameter of an orthotropic disc with speed

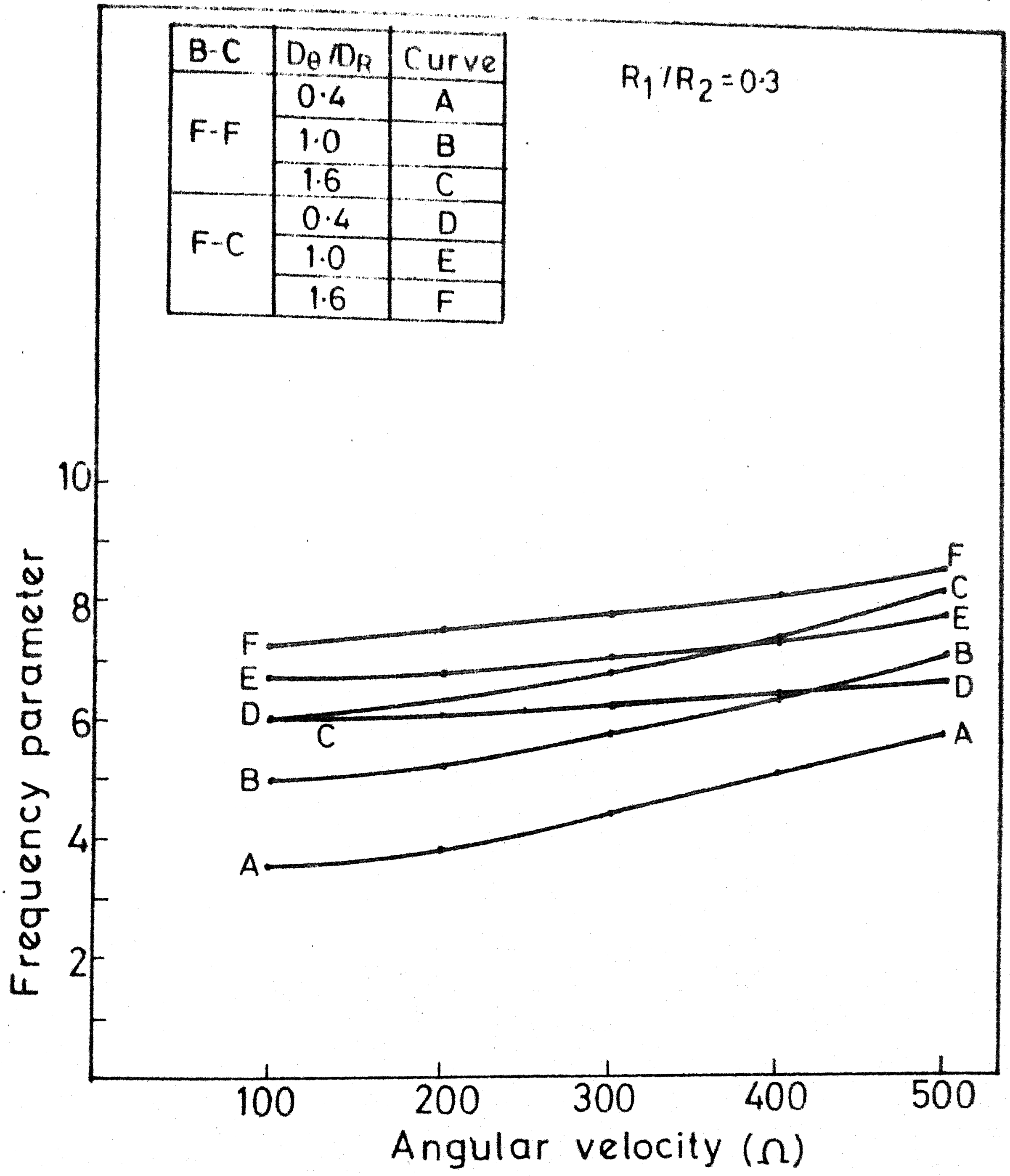


Fig.15 Variation of frequency parameter of an orthotropic disc with speed

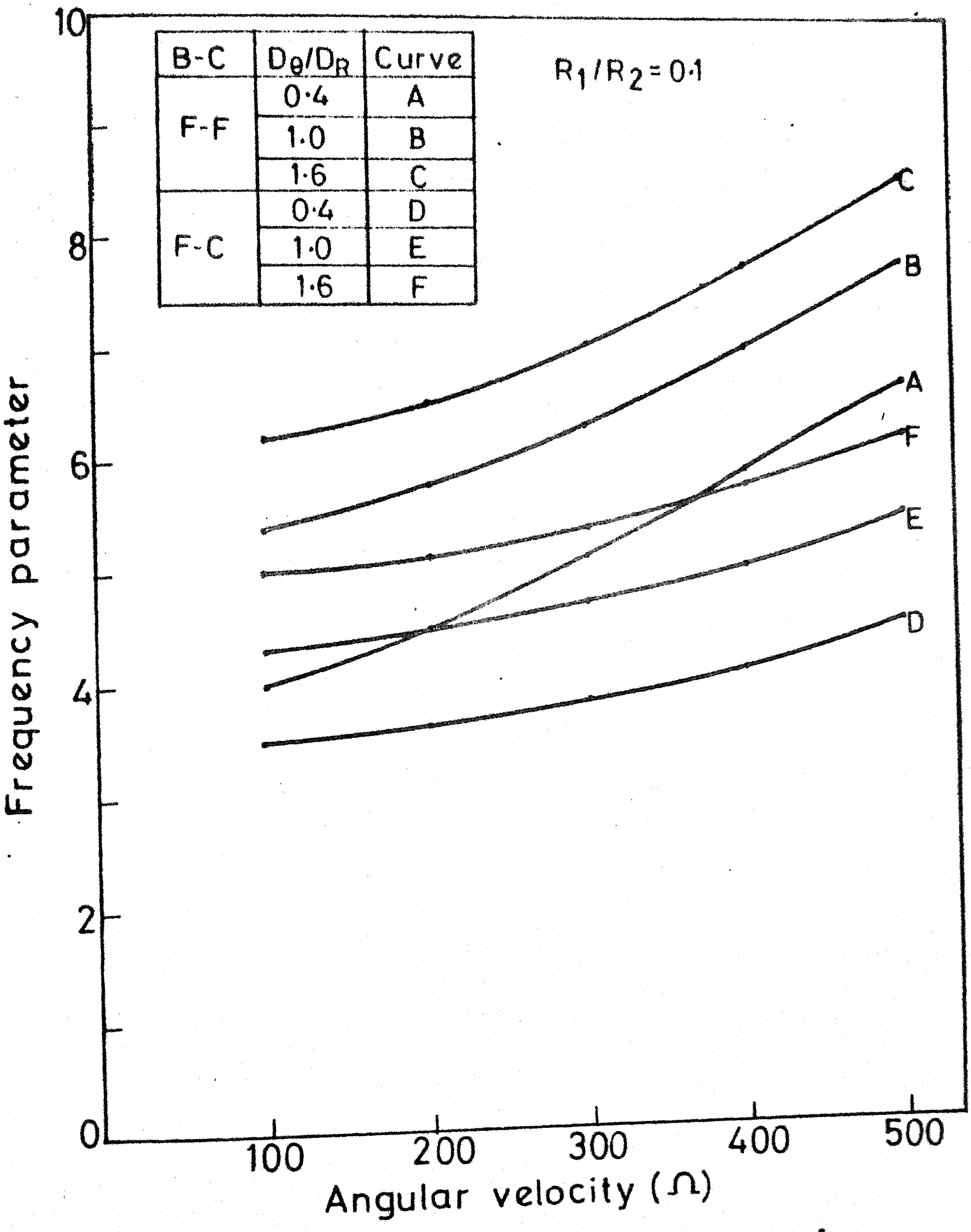


Fig.16 Variation of frequency parameter of an orthotropic disc with speed

Variation of frequency parameter $\lambda R_2^2(h/D_r)^{1/2}$ with angular velocity Ω for different radii ratio and D_θ/D_r ratio

$D_r = 180.00$ $D_{r\theta}/D_r = 0.35$ $\nu_\theta = 0.3$ Ne = 12

B-C	R_1/R_2	D_θ/D_r	Angular velocity Ω				
			100	200	300	400	500
		0.4	3.46	3.61	3.86	4.16	4.52
		1.0	3.30	4.48	4.75	5.11	5.51
		1.6	4.99	5.17	5.44	5.79	6.19
	*Ne=19	0.4	6.04	6.16	6.32	6.57	6.85
F-C	0.3	1.0	6.71	6.87	7.12	7.47	7.88
		1.6	7.32	7.51	7.83	8.24	8.72
(n,m)		0.4	12.49	12.56	12.70	12.88	13.10
0,0	0.5	1.0	13.06	13.18	13.38	13.65	13.99
		1.6	13.60	13.75	14.00	14.35	14.77
		0.4	4.02	4.48	5.15	5.97	6.88
	0.1	1.0	5.43	5.79	6.35	7.06	7.87
		1.6	6.21	6.55	7.07	7.73	8.50
F-F	*Ne=19	0.4	3.47	3.87	4.48	5.13	5.89
	0.3	1.0	5.02	5.35	5.84	6.47	7.17
		1.6	5.90	6.26	6.80	7.4	8.26
(n,m)		0.4	2.90	3.32	3.88	4.49	5.09
0,2	0.5	1.0	4.40	4.77	5.28	5.87	6.48
		1.6	5.27	5.67	6.24	6.91	7.61

Table 13

In Fig. 17 inplane stresses from F.E.M. and exact solutions are plotted. It is observed that convergence is not so rapid as it was for vibration of plates. Hence higher number of elements (25) are required for better accuracy in case of rotating discs. However since stiffness due to inplane stresses forms a part of the overall stiffness, at low speeds, with lower number of elements (12) frequency parameter α is very close to the values obtained using the inplane stresses from exact solution. But it is obvious to go on increasing the number of elements till the values become almost stationary since convergence depends on other factors like boundary conditions and radii ratio also.

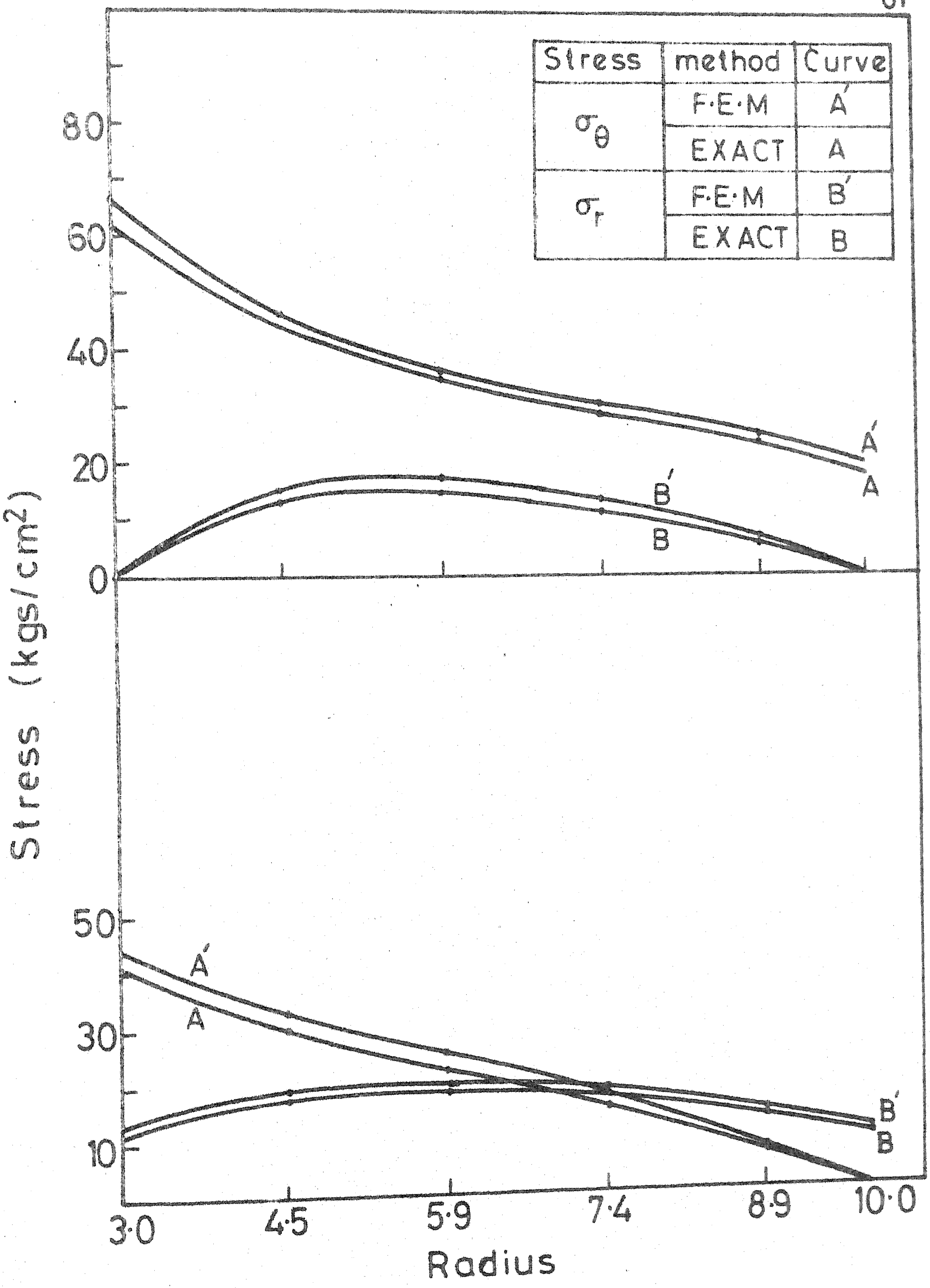


Fig.17 Distribution of inplane stresses (σ_r and σ_θ) along the radius for a rotating disc

CHAPTER - VI

CONCLUSIONS

6.1 CONCLUSIONS

Using semianalytical method vibration analysis of circular and annular plates for various boundary conditions has been done, both for isotropic and orthotropic plates. In the present analysis the problem of material singularity occurring in orthotropic plates is avoided by assuming a small hole at the centre and even then the answers are in good agreement with that of solid plates. The same element has also been used for the study of dynamic behaviour of annular plates with varying thickness. In case of orthotropic plates it has been observed in all cases that the frequency parameter is mainly dependent upon the regidity modulus in radial direction compared to the material properties in other directions. It is because of this reason that when the frequency parameter is determined for a constant D_r with varying D_θ frequency parameter does not vary much. The conclusion to be drawn here is that, by providing sufficient reinforcement in radial direction (polar-orthotropic plates), frequency parameter as high as that for an

isotropic material can be obtained and these orthotropic plates are prefered due to their higher stiffness to weight ratio.

In case of rotating discs the influence of D_θ is more significant. So, for a rotating disc with constant D_r, when D_θ/D_r is increased, it is found that frequency parameter increases more rapidly than that of a stationary plate. This is due to the obvious reason that inplane stresses increase as the regidity modulus D_θ increases.

6.2 SCOPE FOR FUTURE WORK

Since the semianalytical method is quite powerful giving very good results (required for all engineering purposes) further application of the method should be directed towards practical problems like rotating discs with periferal load (turbine disc with blades) and thermal stresses.

Secondly in case of orthotropic plates with varying thickness exact results are not available. So it is necessary to determine the frequency parameter through experiments and compare with the finite element method results.

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APPENDIX-A

BEAM SHAPE FUNCTIONS ARE AS FOLLOWS [23]

$$N_{1x} = \frac{1}{2} - \frac{3}{4}\xi + \frac{1}{4}\xi^{3}$$

$$N_{2x} = \left(\frac{1}{4} - \frac{1}{4}\xi - \frac{1}{4}\xi^{2} + \frac{1}{4}\xi^{3}\right) L_x$$

$$N_{3x} = \frac{1}{2} + \frac{3}{4}\xi - \frac{1}{4}\xi^{3}$$

$$N_{4x} = \left(-\frac{1}{4} - \frac{1}{4}\xi + \frac{1}{4}\xi^{2} + \frac{1}{4}\xi^{3}\right) L_x$$

$$N_{1y} = \frac{1}{2} + \frac{3}{4}\eta - \frac{\eta^{3}}{4}$$

$$N_{2y} = \left(-\frac{1}{4} - \frac{1}{4}\eta + \frac{1}{4}\eta^{2} + \frac{1}{4}\eta^{3}\right) L_y$$

$$N_{3y} = \frac{1}{2} - \frac{3}{4}\eta + \frac{1}{4}\eta^{3}$$

$$N_{4y} = \left(\frac{1}{4} - \frac{1}{4}\eta - \frac{1}{4}\eta^{2} + \frac{1}{4}\eta^{3}\right) L_y$$

APPENDIX-B

B.1 [R] MATRIX OF EQN.(3.2) IS [11]

$$[R] = \begin{bmatrix} \frac{a^2(a-b)}{l^3} & \frac{a^2 b}{l^2} & \frac{b^3(3a-b)}{l^3} & \frac{a\ b^2}{l^2} \\ \frac{6a\ b}{l^3} & \frac{-a(a+2b)}{l^2} & \frac{-6a\ b}{l^3} & \frac{-b(2a+b)}{l^2} \\ \frac{-3(a+b)}{l^3} & \frac{2a+b}{l^2} & \frac{3(a+b)}{l^3} & \frac{(a+2b)}{l^2} \\ \frac{2}{l^3} & \frac{-1}{l^2} & \frac{-2}{l^3} & \frac{-1}{l^2} \end{bmatrix}$$

where $l = a - b$

B.2 COEFFICIENTS OF $[k]^{(e)}$ OF EQN. (3.8)

$$k_{11} = (D_\theta\ m^4 + 4\ D_{r\theta}\ m^2)\ P_{-3}$$

$$k_{12} = D_\theta\ (m^4 - m^2)\ P_{-2}$$

$$k_{13} = (-2\nu_\theta\ D_r\ m^2 + D_\theta\ (m^4 - 2m^2) - 4\ D_{r\theta}\ m^2)\ P_{-1}$$

$$k_{14} = (-6\nu_\theta D_r\ m^2 + D_\theta\ (m^4 - 3m^2) - 8\ D_{r\theta}\ m^2)\ P_o$$

$$k_{22} = D_\theta\ (m^2 - 1)^2\ P_{-1}$$

$$k_{23} = (-2\nu_\theta\ D_r\ (m^2 - 1) + D_\theta\ (m^2 - 1)\ (m^2 - 2))\ P_o$$

$$k_{24} = (-6\nu_\theta\ D_r\ (m^2 - 1) + D_\theta\ (m^4 - 4m^2 + 3))\ P_1$$

$$k_{33} = (4\ D_r - 4\nu_\theta\ D_r\ (m^2 - 2) + D_\theta\ (m^2 - 2)^2 + 4\ D_{r\theta} m^2) P_1$$

$$k_{34} = (12D_r - 6\nu_\theta D_r(m^2 - 2) - 2\nu_\theta D_r (m^2 - 3)$$

$$+ D_\theta (m^2 - 3)(m^2 - 2) + 8D_{r\theta} m^2) P_2$$

$$k_{44} = (36D_r - 12\nu_\theta D_r (m^2 - 3) + D_\theta (m^2 - 3)^2$$

$$+ 16D_{r\theta} m^2) P_3$$

where

$$P_i = \int_b^a h^3 r^i \, dr$$

For linearly varying thickness plate i.e.

$$h = \alpha + \beta r$$

these constants are

$$P_{-3} = \alpha^3 (a^2-b^2)/2ab + \beta^3(a-b) + 3\alpha^2\beta (a-b)/ab$$

$$+ 3\alpha \beta^2 \log (a/b)$$

$$P_{-2} = \alpha^3 (a-b)/ab + \beta^3 (a^2-b^2)/2 + 3\alpha^2\beta \log(a/b)$$

$$+ 3\alpha \beta^2 (a-b)$$

$$P_{-1} = \alpha^3 \log (a/b) + \beta^3 (a^3-b^3)/3 + 3\alpha^2 \beta (a-b)$$

$$+ 3\alpha \beta^2 (a^2-b^2)/2$$

$$P_o = \alpha^3 (a-b) + \beta^3 (a^4-b^4)/4 + 3\alpha^2\beta (a^2-b^2)/2$$

$$+ 3\alpha \beta^2 (a^3-b^3)/3$$

$$P_1 = \alpha^3 (a^2-b^2)/2 + \beta^3 (a^5-b^5)/5 + 3\alpha^2 \beta (a^3-b^3)/3$$

$$+ 3\alpha \beta^2 (a^4-b^4)/4$$

$$P_2 = \alpha^3 (a^3-b^3)/3 + \beta^3 (a^6-b^6)/6 + 3\alpha^2 \beta (a^4-b^4)/4 + 3\alpha \beta^2 (a^5-b^5)/5$$

$$P_3 = \alpha^3 (a^4-b^4)/4 + \beta^3 (a^7-b^7)/7 + 3\alpha^2 \beta (a^5-b^5)/5 + 3\alpha \beta^2 (a^6-b^6)/6$$

where $\alpha = \dfrac{h_1 a - h_2 b}{a - b}$, $\beta = \dfrac{h_2 - h_1}{a - b}$

These relations can be used for uniform plate by putting $\beta = o$ in the above equations.

B.3 COEFFICIENTS OF $[m]^{(e)}$ OF EQN. (3.10)

$$[m]^{(e)} = \begin{bmatrix} Q_1 & Q_2 & Q_3 & Q_4 \\ & Q_3 & Q_4 & Q_5 \\ \text{Sym.} & & Q_5 & Q_6 \\ & & & Q_7 \end{bmatrix}$$

where

$$Q_i = \rho \int_b^a h\, r^i\, dr$$

For a plate with linearly varying thickness i.e. $h = \alpha + \beta r$, these coefficients are

$$Q_1 = \rho (\alpha (a^2-b^2)/2 + \beta (a^3-b^3)/3)$$

$$Q_2 = \rho (\alpha (a^3-b^3)/3 + \beta (a^4-b^4)/4)$$

$$Q_3 = \rho (\alpha (a^4-b^4)/4 + \beta (a^5-b^5)/5)$$

$$Q_4 = \varsigma \ (\alpha \ (a^5-b^5)/5 + \beta \ (a^6-b^6)/6)$$

$$Q_5 = \varsigma \ (\alpha \ (a^6-b^6)/6 + \beta \ (a^7-b^7)/7)$$

$$Q_6 = \varsigma \ (\alpha \ (a^7-b^7)/7 + \beta \ (a^8-b^8)/8)$$

$$Q_7 = \varsigma \ (\alpha \ (a^8-b^8)/8 + \beta \ (a^9-b^9)/9)$$

These coefficients can be used for uniform thickness plate by substituting $\beta = o$ in the above equations.

APPENDIX-C

C.1 COEFFICIENTS OF $[k_p]^{(e)}$ OF EQN. (4.2)

$$[k_p]^{(e)} = \int_b^a [\sigma_r \{s_{,r}\} [s_{,r}] + m^2 \sigma_\theta / r^2 \{s\} [s]] \, h \, r \, dr$$

For a uniform thickness plate

$$[k_p]^{(e)} = \begin{bmatrix} Q_{-1} & Q_o & Q_1 & Q_2 \\ & P_1+Q_1 & 2P_2+Q_3 & 3P_3+Q_3 \\ \text{Sym} & & 4P_3+Q_3 & 6P_4+Q_4 \\ & & & 9P_5+Q_5 \end{bmatrix}$$

where

$$Q_{-1} = h\,(D_1 \log (a/b) + D_2\,(a-b))$$

$$Q_o = h\,(D_1\,(a-b) + D_2\,(a^2-b^2)/2)$$

$$Q_1 = h\,(D_1\,(a^2-b^2)/2 + D_2\,(a^3-b^3)/3)$$

$$Q_2 = h\,(D_1\,(a^3-b^3)/3 + D_2\,(a^4-b^4)/4)$$

$$Q_3 = h\,(D_1\,(a^4-b^4)/4 + D_2\,(a^5-b^5)/5)$$

$$Q_4 = h\,(D_1\,(a^5-b^5)/5 + D_2\,(a^6-b^6)/6)$$

$$Q_5 = h\,(D_1\,(a^6-b^6)/6 + D_2\,(a^7-b^7)/7) \quad \text{and}$$

$$P_1 = h\,m^2\,(E_1\,(a^2-b^2)/2 + E_2\,(a^3-b^3)/3)$$

$$P_2 = h\,m^2\,(E_1\,(a^3-b^3)/3 + E_2\,(a^4-b^4)/4)$$

$$P_3 = h\,m^2\,(E_1\,(a^4-b^4)/4 + E_2\,(a^5-b^5)/5)$$

$$P_4 = h\,m^2\,(E_1\,(a^5-b^5)/5 + E_2\,(a^6-b^6)/6)$$

$$P_5 = h\cdot m^2\,(E_1\,(a^6-b^6)/6 + E_2\,(a^7-b^7)/7)$$

where

$$D_1 = \frac{\sigma_{\theta_1} - \sigma_{\theta_2}}{a - b}, \quad D_2 = \frac{\sigma_{\theta_2} - \sigma_{\theta_1}}{a - b}$$

$$E_1 = \frac{\sigma_{r_1} - \sigma_{r_2}}{a - b}, \quad E_2 = \frac{\sigma_{r_1} - \sigma_{r_2}}{a - b}$$

C.2 COEFFICIENTS OF $[k_i]^{(e)}$ OF EQN. (4.9)

$$k_{i_{11}} = C_5 \log (a/b)$$

$$k_{i_{12}} = (C_5+C_6)\,(a-b)$$

$$k_{i_{13}} = (2C_6 + C_5)\,(a^2-b^2)/2$$

$$k_{i_{14}} = (3C_6 + C_5)\,(a^3-b^3)/3$$

$$k_{i_{22}} = (2C_6 + C_4 + C_5)\,(a^2-b^2)/2$$

$$k_{i_{23}} = (2C_4 + C_5 + 3C_6)\,(a^3-b^3)/3$$

$$k_{i_{24}} = (3C_4 + C_5 + 4C_6)\,(a^4-b^4)/4$$

$$k_{i_{33}} = (4C_4 + C_5 + 4C_6)\,(a^4-b^4)/4$$

$$k_{i_{34}} = (6C_7 + C_5 + 5C_6)\,(a^5-b^5)/5$$

$$k_{i_{44}} = (9C_4 + C_5 + 6C_6)\,(a^6-b^6)/6$$

$$k_{i_{mn}} = k_{i_{nm}}$$

where

$$C_4 = \frac{2\pi\ E_r\ h}{(1-\nu_r\ \nu_\Theta)}$$

$$C_5 = \frac{2\pi\ E_\Theta\ h}{(1-\nu_r\ \nu_\Theta)}$$

$$C_6 = C_4\ \nu_\Theta\ .$$